Calibrated geometry in manifolds of exceptional holonomy

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Alla mia famiglia,
semplicemente grazie.

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Abstract

In this thesis, we discuss some aspects of calibrated geometry in manifolds of exceptional holonomy. Manifolds of exceptional holonomy are Riemannian manifolds that are endowed with one of the following additional structures: a torsion-free G_2-structure or a torsion-free Spin(7)-structure. G_2 manifolds admit two special families of calibrated, hence volume minimizing, submanifolds: associative 3-folds and coassociative 4-folds. Spin(7) manifolds admit only one family of calibrated submanifolds: Cayley 4-folds. Understanding the geometry of such calibrated submanifolds is one of the key challenges in the study of manifolds with exceptional holonomy.

After recalling some basic notion on calibrated geometry and manifolds of exceptional holonomy, we define calibrated fibrations, and we prove a rigidity result for these objects under some linear condition.

Then, we describe the construction of two Cayley fibrations in the Bryant–Salamon Spin(7) manifold using a cohomogeneity one method. These are the first explicit examples of Cayley fibrations in a non-flat Spin(7) manifold and the fibres provide new examples of Cayley submanifolds.

Finally, we study the geometry of calibrated submanifolds in G_2 manifolds that admit $\mathbb{T}^2 \times \mathrm{SU}(2)$-symmetry. We apply our results to $\mathbb{C}^3 \times S^1$, to the Bryant–Salamon manifolds and to the manifolds recently constructed by Foscolo–Haskins–Nordström, where our analysis gives new large families of $\mathbb{T}^2$-invariant associatives. This is based on joint work with B. Aslan.

Contents

Chapter 1

Introduction

1.1 Motivation

1.1.1 Riemannian holonomy and Berger's list

Given a connected Riemannian n-manifold (M^n, g) and a point $x \in M$, the Levi-Civita connection induces the group:

$$\mathrm{Hol}_x(g) := \{P_\gamma \subset \mathrm{O}(T_xM) : \gamma \text{ is a loop based at } x\},$$

where P_γ denotes the parallel transport along γ. It is fairly easy to see that such a group is a Lie group and it is independent from x up to conjugation. Hence, it makes sense to call it the Riemannian holonomy group of (M, g), $\mathrm{Hol}(g)$, and to regard it as a Lie subgroup of $\mathrm{O}(n, \mathbb{R})$, defined up to conjugation.

A natural question that arises is the following:

Question 1.1.1. What are the subgroups of $\mathrm{O}(n, \mathbb{R})$ that can appear as the Riemannian holonomy group of some Riemannian manifold (M^n, g)?

Before approaching Question 1.1.1, we need the following observations. Firstly, it is sensible to recast Question 1.1.1 assuming M^n to be simply-connected. Indeed, $\mathrm{Hol}(g)$ naturally encodes information on the fundamental group of M, $\pi_1(M)$, as the loops in the definition of holonomy do not need to be homotopic to the constant path. The formal way around it is by considering the restricted holonomy group:

$$\mathrm{Hol}^0_x(g) := \{P_\gamma \subset \mathrm{O}(T_xM) : \gamma \text{ is a loop based at } x \text{ homotopic to the constant path}\},$$

which can be shown to be a normal Lie subgroup of $\mathrm{Hol}_x(g)$ and that coincides with the identity component of $\mathrm{Hol}_x(g)$. The groups $\mathrm{Hol}(g), \mathrm{Hol}^0(g)$ and $\pi_1(M)$ are related by the group homomorphism:

$$\begin{aligned}\phi : \pi_1(M) &\to \mathrm{Hol}(g)/\,\mathrm{Hol}^0(g)\\ [\gamma] &\mapsto P_\gamma \cdot \mathrm{Hol}^0(g).\end{aligned}$$

Secondly, it is straightforward to verify that, if $(M^n, g) = (M_1 \times M_2, g_1 \times g_2)$ then, $\mathrm{Hol}(g) = \mathrm{Hol}(g_1) \times \mathrm{Hol}(g_2)$. More surprisingly, the converse holds locally, i.e., if the Riemannian holonomy group of (M^n, g) is reducible as a representation on $\mathbb{R}^n$, then, M^n is locally isometric to a Riemannian product. As we are interested in the "building blocks" of the holonomy group, we will assume in Question 1.1.1 that (M^n, g) is irreducible, i.e. that it is not locally isometric to a Riemannian product.

Finally, we will restrict ourselves to Riemannian manifolds which are nonsymmetric, i.e. that are not locally isometric to a Riemannian symmetric space. The reason is that the holonomy group of a simply-connected Riemannian symmetric space can be easily deduced from its structure. Moreover, symmetric spaces were completely classified by Cartan in [20, 21] (cfr. [13, Chapter 7.H] for a list of this classification).

We refer the reader to [13, 46, 52, 53] for further details on the Riemannian holonomy group, its properties and the assumptions that we have discussed.

We are now ready to answer Question 1.1.1.

Theorem 1.1.2 (Berger [12])**.** *Let (M^n, g) be a simply-connected, irreducible, nonsymmetric Riemannian manifold. Then, one of the following is satisfied:*

1. (Generic case) $\mathrm{Hol}(g) = \mathrm{SO}(n)$,
2. (Kähler case) $\mathrm{Hol}(g) = \mathrm{U}(m) \subset \mathrm{SO}(n)$, *where $n = 2m$ for some $m \in \mathbb{N}$* ,
3. (Calabi–Yau case) $\mathrm{Hol}(g) = \mathrm{SU}(m) \subset \mathrm{SO}(n)$, *where $n = 2m$ for some $m \in \mathbb{N}$,*
4. (Hyperkähler case) $\mathrm{Hol}(g) = \mathrm{Sp}(m) \subset \mathrm{SO}(n)$, *where $n = 4m$ for some $m \in \mathbb{N}$,*
5. (Quaternionic Kähler case) $\mathrm{Hol}(g) = \mathrm{Sp}(m) \cdot \mathrm{Sp}(1) \subset \mathrm{SO}(n)$, *where $n = 4m$ for some $m \in \mathbb{N}$,*
6. (G_2 case) $\mathrm{Hol}(g) = \mathrm{G}_2 \subset \mathrm{SO}(7)$,
7. ($\mathrm{Spin}(7)$ case) $\mathrm{Hol}(g) = \mathrm{Spin}(7) \subset \mathrm{SO}(8)$.

Since G_2 and $\mathrm{Spin}(7)$ do not come in a countable family depending on the dimension of the Riemannian manifold, these groups are usually referred to as the exceptional holonomy groups, and will be the central objects of this thesis. Intuitively, the reason behind this phenomenon boils down to the octonionic nature of G_2 and $\mathrm{Spin}(7)$ and the fact that putting an $\mathbb{O}$-module structure on $\mathbb{O}^n$ is not really a sensible idea, as $\mathbb{O}$ is a non-associative normed division algebra.

Observe that, a priori, not all the groups in Theorem 1.1.2 need to be the Riemannian holonomy of some (M, g). For instance, the original Berger's list contained $\mathrm{Spin}(9) \subset \mathrm{SO}(16)$ as well. However, Alekseevskii [4] and Brown–Gray [17] ruled out this case by showing, independently, that Riemannian manifolds with holonomy $\mathrm{Spin}(9)$ need to be symmetric. All the remaining elements of Berger's list are attained, i.e. they are the Riemannian holonomy group of some Riemannian manifold (M, g). For further information on the Berger's list and examples with a given holonomy, we direct the reader to [13, 46, 67] and references therein.

1.1.2 The holonomy principle and calibrated geometry

Assume that (M, g) is a Riemannian manifold with holonomy some given Lie group G. The definition of Riemannian holonomy group does not necessarily enlighten on how such condition shapes the geometry of M.

Question 1.1.3. Can we translate the condition on $\mathrm{Hol}(g)$ to a more tangible property of the Riemannian manifold?

The answer to this question is given by the holonomy principle. Roughly speaking, it says that $\mathrm{Hol}(g)$ determines the parallel tensors of M and vice versa.

Proposition 1.1.4 (Holonomy principle)**.** *Let (M, g) be a Riemannian manifold and let $E = \otimes^k TM \otimes^l T^*M$ be endowed with the natural connection induced from the Levi-Civita connection. If $S \in \Gamma(E)$ is such that $\nabla S \equiv 0$, then, S_p is fixed by the natural extension of the action of $\mathrm{Hol}(g)$ on E_p, for every $p \in M$. Conversely, if $A \in E_p$ is fixed by the natural extension of the action of $\mathrm{Hol}(g)$, then, there exists a unique $S \in \Gamma(E)$ such that $\nabla S \equiv 0$ and $S_p = A$.*

Corollary 1.1.5. *Let (M, g) be a simply-connected, irreducible, nonsymmetric Riemannian manifold and let $p \in M$ fixed. If $G \subset \mathrm{SO}(T_pM)$ is the subgroup that fixes S_p for every parallel tensor S, then, $G = \mathrm{Hol}_p(g)$.*

A straightforward consequence of the holonomy principle is that Riemannian manifolds with holonomy in the Berger's list come equipped with parallel (and hence closed) differential forms, which, up to rescaling, can be assumed to have co-mass 1. Differential forms satisfying these conditions are calibrations and determine a special family of volume-minimizing submanifolds: calibrated submanifolds.

Calibrated geometry was introduced by Harvey and Lawson in their seminal work [37], where they also highlighted its connection to Riemannian holonomy. We give further details on calibrated geometry in Section 2.1, where we also sum up the calibrations that arise in manifolds with special holonomy (cfr. Example 2.1.7).

1.1.3 Exceptional geometries and calibrated submanifolds

We now turn our attention to the exceptional Riemannian holonomy groups G_2 and Spin(7). From Section 1.1.2, one can prove that manifolds with G_2 holonomy have a calibrating 3-form, φ, and a calibrating 4-form, $*\varphi$, which is simply the Hodge dual of φ. Submanifolds calibrated by φ are called associative submanifolds, while the ones calibrated by $*\varphi$ are called coassociative submanifolds. Manifolds with Spin(7) holonomy only have one calibrating 4-form, Φ, whose calibrated submanifolds are called Cayleys.

As mentioned in Section 1.1.1, the groups G_2 and Spin(7) are attained as the Riemannian holonomy group of some Riemannian manifold. Indeed, Bryant provided the first incomplete examples in [18], Bryant–Salamon constructed the first complete non-compact ones in [19] and Joyce settled the compact case in [43–45]. Since then, much effort has been spent to construct new G_2 and Spin(7) manifolds and, now, we have a large variety of complete manifolds with such holonomy groups (see for instance [9, 10, 14, 23–25, 30–32, 34, 47, 56] and many more). Of particular interest are the G_2 manifolds recently constructed by Foscolo–Haskins–Nordström in [32] (cfr. Section 2.2.5). Indeed, this family extends all the previously known (apart from [31]) complete non-compact examples and they are explicit up to solving a system of ODEs.

A different story holds for complete calibrated submanifolds, where only a handful of them are known compared to the number of G_2 and Spin(7) manifolds. Here is a brief description of all the previously known examples. In the local model, $\mathbb{R}^7$ and $\mathbb{R}^8$, calibrated submanifolds were constructed by Harvey–Lawson and Lotay assuming cohomogeneity one symmetry [37, 59, 61], by Lotay assuming the submanifold to be ruled [60] and by Ionel–Karigiannis–Min-Oo assuming the submanifold to be a vector subbundle [42]. The first non-trivial examples of calibrated submanifolds in a non-flat manifold of exceptional holonomy were constructed on the Bryant–Salamon manifolds of topology $\Lambda^2_-(S^4)$, $\Lambda^2_-(\mathbb{CP}^2)$ and $\not{S}_-(S^4)$ by Karigiannis–Min-Oo [50], extending [42] to the non-flat setting. On the Bryant–Salamon manifolds of topology $\Lambda^2_-(X)$ cohomogeneity one techniques were used by Kawai [51] and Karigiannis–Lotay [49] to produce coassociative submanifolds. For what concerns compact manifolds, the only closed calibrated submanifolds are described in [11, 23, 28, 36, 43–45].

One of the main goals of this thesis is to produce new examples of calibrated submanifolds in non-compact manifolds of exceptional holonomy. In particular, we construct large families of Cayley submanifolds on the Spin(7) Bryant–Salamon manifolds (cfr. Chapter 4) and large families of associative submanifolds on each G_2 manifold constructed by Foscolo–Haskins–Nordström (cfr. Chapter 5). Often, the calibrated submanifolds that we construct form a calibrated fibration, which is, roughly speaking, a fibre bundle with calibrated fibres up to a measure zero set (cfr. Definition 3.1.5). These objects have been widely studied both because of their connection to physics (cfr. Section 1.1.4) and because one could hope to construct new manifolds with exceptional holonomy from them [2,8,27,49,57]. In this direction, Donaldson [27] studied coassociative fibrations and Cayley fibrations under an "adiabatic limit", i.e. when the volume of the fibres is sent to zero. In Chapter 3, we consider a sort of opposite procedure, i.e. we study coassociative and Cayley fibrations with some natural linear structure.

1.1.4 Exceptional holonomy in mathematical physics

Apart from being interesting mathematical objects, manifolds with exceptional holonomy have also drawn the attention of mathematical physicists and are now widely studied by that community as well.

A first reason for their interest is that a Riemannian manifold, (M, g), with Riemannian holonomy G_2 and Spin(7) (but also $SU(m)$, $Sp(m)$) needs to have vanishing Ricci tensor [4], and Ricci-flat manifolds are (positive definite) solutions of Einstein's field equations in vacuum, with vanishing cosmological constant. To highlight the importance of this phenomenon, we remark that all the known examples of Ricci-flat compact nonsymmetric Riemannian manifolds have holonomy $SU(m)$, $Sp(m)$, G_2 or Spin(7). More details on Ricci-flat manifolds can be found in [13] and references therein.

A second connection between exceptional holonomy and mathematical physics comes from string theory and the relative generalizations. Roughly speaking, supersymmetric string theory (M-theory, F-theory) claims that the universe should be a 10-dimensional (11-dimensional, 12-dimensional) fibre bundle over an Einstein space-time and the fibres should be compact and very small manifolds with holonomy SU(3) (G_2, Spin(7)). This additional dimensions, parametrizing the manifold with special holonomy, should correspond to the space where "quantum phenomena" occur.

One of the most important conjectures string theorists are interested in is mirror symmetry. It is outside of the scope of this thesis to give a precise account of this conjecture. An introduction to string theory and mirror symmetry for "dummies", as the author claims (but still quite out of reach for me), can be found in [46, Chapter 9].

What matters is that a geometrical interpretation of mirror symmetry was proposed by Strominger–Yau–Zaslow in [70] for supersymmetric string theory and, afterwards further generalized to M-thory and F-theory by Gukov–Yau–Zaslow in [35]. Their idea was to interpret "mirror phenomena" in terms of calibrated fibrations, as defined in the previous section. Even though the SYZ conjecture and generalizations have received the tireless attentions of both mathematicians and physicists (cfr. for instance [1, 3]), it still looks inaccessible at the moment.

1.2 Overview of the thesis

1.2.1 Chapter 2: calibrated geometry and exceptional holonomy

In Chapter 2, we cover the preliminaries for the rest of the thesis. As a first step, we recall the definition of calibration, of calibrated (current) submanifold and we prove that calibrated submanifolds are volume-minimizing in their homology class. Moreover, we provide the list of calibrated submanifolds that arise in manifolds of special holonomy via Proposition 1.1.4.

Afterwards we turn our attention to manifolds with holonomy group contained in G_2. Using Corollary 1.1.5, we characterize them as the 7-manifolds admitting a particular closed and co-closed 3-form, the G_2-structure, and we show how it induces a cross product on the tangent bundle. Then, we turn our attention to associative and coassociative submanifolds, i.e. the submanifolds calibrated by one of the two characterizing forms, respectively. After providing some basic properties of these objects, we recall some machinery of geometric measure theory for currents with symmetry. We conclude our discussion on G_2 manifolds with a brief description of the Bryant–Salamon manifolds of topology $S^3 \times \mathbb{R}^4$ [19] and of the Foscolo–Haskins–Nordström manifolds [32].

Similarly to the G_2 setting, we use Corollary 1.1.5 to characterize manifolds with holonomy contained in Spin(7) as the 8-manifolds admitting a particular closed 4-form, the Spin(7)-structure, which induces a triple cross product on the tangent bundle. Subsequently, we give a short introduction to Cayley submanifolds, i.e. the submanifolds calibrated by the Spin(7)-structure, and we give a concise description of the Spin(7) Bryant–Salamon manifolds.

We conclude this chapter recalling the theory of multi-moment maps introduced by Madsen and Swann in [62, 63]. Multi-moment maps are natural extensions of symplectic geometry's moment maps to manifolds that possess a generic closed form. The idea is to take generators of the automorphism group and contract them with the given closed form to reduce its degree to 1. Now, if the resulting 1-form is exact, it can be integrated to a function: a multi-moment map. Since it is not true that for all generators the induced 1-form needs to be exact, Madsen and Swann introduced the notion of kth Lie kernel to overcome this issue.

1.2.2 Chapter 3: calibrated fibrations and linear calibrated vector bundles

The first part of Chapter 3 is devoted to the mathematical notion of calibrated fibrations. The "naive" way to define calibrated fibrations is by assuming that the manifold is a locally trivial fibre bundle with calibrated fibres. After recalling Baraglia's nonexistence result for locally trivial coassociative fibrations [8], we provide a more general definition of calibrated fibrations (cfr. Definition 3.1.5), which is inspired by the work of Karigiannis–Lotay [49]. This second definition allows the fibres to be singular and to intersect.

In [27], Donaldson characterized locally trivial coassociative fibrations and Cayley fibrations (cfr. Proposition 3.1.2 and Proposition 3.1.3) as fibre bundles endowed with an Ehresmann connection and suitable tensors related by a system of PDEs. He studied such a system by taking an "adiabatic limit", i.e. letting the size of the fibres approach zero. Under such a procedure, the system de-couples and can be solved completely. Via a perturbation argument, he used the solutions of the adiabatic system to obtain solutions of the original problem.

As a dual approach, if we let the size of the fibres explode, one should obtain in the limit a locally trivial coassociative (Cayley) fibration with a compatible vector bundle and G_2-structure (Spin(7)-structure). In the remaining part of this chapter we take first steps towards a classification of these "linear coassociative fibrations" ("linear Cayley fibrations"). In particular, we show that under some isotropic condition the only linear coassociative (Cayley) fibrations are deformations of the Bryant–Salamon manifolds described in Section 2.2.4 and Section 2.3.3 (cfr. Theorem 3.2.6 and Theorem 3.3.5).

1.2.3 Chapter 4: Cayley fibrations in the Bryant–Salamon $\mathrm{Spin}(7)$ manifolds

In Chapter 4, we describe the first explicit examples of Cayley fibrations in a non-flat Spin(7) manifold: the Bryant–Salamon Spin(7) manifold $(\not{S}_-(S^4), \Phi_c)$. This chapter is based on the author's paper [71].

The main technique for the construction is a cohomogeneity one method, which reduces the problem to a system of ODEs in the orbit space. The actions that we consider are the

3-dimensional subgroups of the automorphism group $\mathrm{Aut}(M, \Phi_c) \cong \mathrm{Sp}(2) \times \mathrm{Sp}(1)$ that do not sit diagonally in it (cfr. Section 2.3.3.3):

$$G \times \mathrm{Id}_{\mathrm{Sp}(1)} \subset \mathrm{Sp}(2) \times \mathrm{Sp}(1), \qquad \mathrm{Id}_{\mathrm{Sp}(2)} \times \mathrm{Sp}(1) \subset \mathrm{Sp}(2) \times \mathrm{Sp}(1),$$

where G is the double lift to $\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$ of one of the following subgroups of $\mathrm{SO}(5)$:

$$\mathrm{SO}(3) \times \mathrm{Id}_2, \quad \mathrm{Sp}(1) \times \mathrm{Id}_1, \quad \mathrm{SO}(3) \text{ acting irreducibly on } \mathbb{R}^5.$$

In each section, we deal with one of the aforementioned actions, sarting with $\mathrm{Id}_{\mathrm{Sp}(2)} \times \mathrm{Sp}(1)$ in Section 4.1, where the fibration is the natural vector bundle projection: $\pi : \not{S}_-(S^4) \to S^4$.

Section 4.2 is the first non-trivial case, where we consider the action induced from $\mathrm{SO}(3) \times \mathrm{Id}_2$. The first crucial idea is to parametrize the sphere S^4 according with the splitting $\mathbb{R}^3 \oplus \mathbb{R}^2$, so that $\mathrm{SO}(3) \times \mathrm{Id}_2$ only acts on the first component. This parametrization induces a trivialization of the bundle $\not{S}_-(S^4)$. After computing how the $\mathrm{Sp}(1)$-action lifts in this trivialization (Section 4.2.3), we notice that the Hopf fibration map on the fibres is compatible with the action and, hence, we reparametrize the fibres according to it. We are then able to find the ODE system for Cayley submanifolds (Section 4.2.7) after a diagonalizing change of frame (Section 4.2.6). The system of ODEs that we obtain is well-behaved and can be completely integrated (Proposition 4.2.18). The geometrical information that we can obtain from the system is derived in Section 2.3.3. In particular, we deduce that the base space of the fibration is S^4 and that the fibres are smooth complete $\mathcal{O}_{\mathbb{CP}^1}(-1)$s, smooth complete $S^3 \times \mathbb{R}$ or conically singular $\mathbb{R}^4$s. A summary of all this is given in Theorem 4.2.20 (Theorem 4.2.21 for the conical case).

In Section 4.3 we deal with the action induced from $\mathrm{Sp}(1) \times \mathrm{Id}_1$. As for the previous case, after appropriate choices of a parametrization (Section 4.3.1, Section 4.3.3, Section 4.3.4) we are able to find a well-behaved system of ODEs. In this case, it is not completely integrable, but reduces to a autonomous system on the plane that we can study via a dynamical system argument. From the ODE system, we deduce that the fibration is parametrized by S^4 and that the fibres are smooth complete $S^3 \times \mathbb{R}$ or smooth complete $\mathbb{R}^4$. The geometry of the Cayley fibration is encapsulated in Theorem 4.3.8 (Theorem 4.3.9 for the conical case).

The remaining group is $\mathrm{Sp}(1)$ induced from the irreducible action of $\mathrm{SO}(3)$ on $\mathbb{R}^5$. In this case, a suitable choice of parametrization is not available, hence, we fail to provide a Cayley fibration (cfr. Section 4.4 for further details).

For each of these actions we compute the associated multi-moment map. Unfortunately, there is no clear interpretation of these maps in terms of the fibration.

1.2.4 Chapter 5: calibrated geometry in G_2 manifolds with $\mathbb{T}^2 \times \mathrm{SU}(2)$-symmetry

Chapter 5, based on a joint work with B. Aslan, is devoted to G_2 manifolds that admit a cohomogeneity two, structure-preserving $\mathbb{T}^2 \times \mathrm{SU}(2)$-action. Before going into the details of it, we recall that the Bryant–Salamon manifolds of topology $\not{S}(S^3)$ and the complete manifolds constructed by Foscolo–Haskins–Nordström have the desired symmetry. Hence, we have a large family of examples falling into this class.

As a first step, we study the geometry of a structure-preserving and cohomogeneity two $\mathbb{T}^2 \times \mathrm{SU}(2)$-action on a G_2 manifold, (M, φ). For the sake of clarity, in this section (and only in this section) we assume that the action is effective. This avoids the technical problem of passing to suitable quotients. We show that the principal stabilizer of the $\mathbb{T}^2 \times \mathrm{SU}(2)$-action is trivial and that there are no exceptional orbits, i.e. there are no points with non-trivial discrete stabilizer. Moreover, the singular part, i.e. the set where the stabilizer is not trivial, further splits into 4 strata, which are characterized by the dimension of the stabilizer. Explicitly, we have:

$$M = M_P \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4,$$

where M_P is the principal part and $\mathcal{S}_i$ are embedded submanifolds characterized by having i-dimensional $\mathbb{T}^2 \times \mathrm{SU}(2)$-stabilizer at each point. The dimension of $\mathcal{S}_4$ is 1, of $\mathcal{S}_3$ is 3, of $\mathcal{S}_2$ is 3 and of $\mathcal{S}_1$ is 5. We conclude the first section by studying the properties of the multi-moment maps associated to $\mathbb{T}^2 \times \mathrm{SU}(2)$, which are ν (relative to φ and $\mathbb{T}^2 \cong \mathbb{T}^2 \times \mathrm{Id}_{\mathrm{SU}(2)}$), θ (relative to φ and $S^1 \times S^1 \subset \mathbb{T}^2 \times \mathrm{SU}(2)$), μ (relative to $*\varphi$ and $\mathbb{T}^2 \times S^1 \subset \mathbb{T}^2 \times \mathrm{SU}(2)$) and η (relative to $*\varphi$ and $\mathrm{Id}_{\mathbb{T}^2} \times \mathrm{SU}(2)$).

In Section 5.2, we focus on the local characterization of G_2 manifolds with $\mathbb{T}^2 \times \mathrm{SU}(2)$-symmetry. Our analysis is based on Madsen and Swann's work in the $\mathbb{T}^2$-case [62]. There, they used Hitchin's flow [41] to locally recover G_2 manifolds with $\mathbb{T}^2$-symmetry from a coherently tri-symplectic four-manifold (see Definition 5.2.2) endowed with a suitable $\mathbb{R}^2$-valued 2-form. Since the enhanced $\mathrm{SU}(2)$-symmetry passes to such a four-manifold, we conclude (Theorem 5.2.9) that G_2 manifolds with $\mathbb{T}^2 \times \mathrm{SU}(2)$-symmetry are locally characterized by two nested system of ODEs (the Hitchin's flow and the one for coherently tri-sympliectic four manifolds with $\mathrm{SU}(2)$-symmetry) and a suitable $\mathbb{R}^2$-valued two form. Note that this system can always be locally solved under some real-analyticity condition.

Afterwards, we turn our attention to cohomogeneity one calibrated submanifolds. In particular, we consider $\mathbb{T}^2 \times \mathrm{Id}_{\mathrm{SU}(2)}$-invariant associatives (Section 5.3), $\mathbb{T}^2 \times S^1$-invariant coassociatives and $\mathrm{Id}_{\mathbb{T}^2} \times \mathrm{SU}(2)$-invariant coassociatives (Section 5.4).

For what concerns finding $\mathbb{T}^2 \times \mathrm{Id}_{\mathrm{SU}(2)}$-invariant associatives, we show that the problem splits with the aforementioned stratification and that μ is a first integral of the induced ODE system. As a consequence of this, together with the slice theorem, we prove that (cfr. Theorem 5.3.5 and Theorem 5.3.9):

- $\mathcal{S}_3 \cup \mathcal{S}_4$ is an associative submanifold,
- $\mathcal{S}_2$ is an associative submanifold,
- $\mathcal{S}_1$ admits a $\mathbb{T}^2$-invariant submersion $F : \mathcal{S}_1 \to S^2$, with associative fibres,
- $\mathbb{T}^2$-invariant associatives project in $B := M_P/(\mathbb{T}^2 \times \mathrm{SU}(2))$ to level sets of $|\mu|$ and they can be recovered from such level sets up to an horizontal lift.

Along the way, we show that the aforementioned associatives are all smooth, and we discuss when they form a calibrated fibration.

It was observed by Madsen and Swann in [64] that $\mathbb{T}^2 \times S^1$-invariant coassociatives are the level sets of some components of the multi-moment maps θ and ν. We observe that these coassociatives project on B to the level sets of ν and, since $(\mu, |\nu|) : B \to \mathbb{R}^2$ is a local diffeomorphism, they form together with the $\mathbb{T}^2$-invariant associatives a local associative/coassociative parametrization of B (Corollary 5.3.10). We conclude our discussion on $\mathbb{T}^3$-invariant associatives by showing that all singular points admit a tangent cone modelled on the Harvey–Lawson cone times $\mathbb{R}$.

Differently from the other cases, SU(2)-invariant coassociatives only exists when φ vanishes on the orbits of the SU(2)-action and do not correspond to level sets of multi-moment maps on B. However, we manage to show, using geometric measure theory that when they exist they are all smooth.

We conclude this chapter applying our results to $\mathbb{C}^3 \times S^1$, the Foscolo–Haskins–Nordström manifolds and the Bryant–Salamon manifolds of topology $\not{S}(S^3)$ (cfr. Section 5.5). In addition, we also extend the (possibly twisted) vector subbundle construction to the Bryant–Salamon manifolds of topology $\not{S}(S^3)$.

Chapter 2

Calibrated geometry and exceptional holonomy

In this chapter, we provide a brief overview of calibrated geometry, manifolds with Riemannian holonomy group G_2, manifolds with Riemannian holonomy group Spin(7) and their relation.

In particular, we review the first definitions and properties of calibrated submanifolds and of manifolds with holonomy G_2 and Spin(7), together with the examples constructed by Bryant and Salamon [19] and by Foscolo, Haskins and Nordström [32]. We recall that G_2 and Spin(7) manifolds admit natural calibrations and calibrated submanifolds, called associatives and coassociatives in the former and Cayleys in the latter [37]. We review some basic properties of such calibrations and calibrated submanifolds.

Finally, we discuss some generalization of the classical notion of moment maps in symplectic geometry due to Madsen and Swann [62, 63]. These objects, called multi-moment maps, will play a crucial role in Chapter 5.

2.1 Calibrated geometry

Let (M, g) be a Riemannian manifold. We recall the definition, due to Harvey and Lawson [37], of calibrations, calibrated currents and calibrated submanifolds.

Definition 2.1.1. A calibration on (M, g) is a k-form, α, on M such that:

- $d\alpha = 0$,
- $M(\alpha) \leq 1$,

where $M(\alpha) := \sup_{x \in M}\{\alpha(X_1, \dots, X_k) : X_i \in T_xM, |X_i| = 1\}$ denotes the co-mass of α.

Given a calibration on a Rimennian manifold (M, g), there is a special class of currents and submanifolds that are determined by the calibration. First, we recall the notion of locally integer rectifiable currents.

Definition 2.1.2. Let $\Gamma_c(M, \Lambda^k T^*M)$ be the space of compactly supported k-forms on M and let $T \in (\Gamma_c(M, \Lambda^k T^*M))^*$ be an element of its topological dual, i.e. a current. The current T is said to be locally integer rectifiable if there exists:

1. a sequence of C^1 oriented k-submanifolds, Σ_i;

2. a sequence of pairwise disjoint closed subsets, $K_i \subset \Sigma_i$;

3. a sequence of positive integers k_i,

such that:

1. $\sum_i k_i \mathcal{H}^k(K_i \cap \Omega) < \infty$, for every Ω compact in M;

2. $T(\omega) = \sum_i k_i \int_{K_i} \omega$ for every $\omega \in \Gamma_c(M, \Lambda^k T^*M)$.

In particular, the support of T is $\overline{\cup_i K_i}$ and at each point $p \in \cup_i K_i$ we can define $\overrightarrow{T}(p) \in \Lambda^k T_p M$ representing the tangent space of the appropriate Σ_i.

We are now ready to define the locally integer rectifiable currents that are determined by the calibration.

Definition 2.1.3. Let (M, g) be a Remannian manifold, and let α be a k-dimensional calibration on it. A k-dimensional locally integer rectifiable current, T, is calibrated by α (also called α-current) if $\langle \alpha_p, \overrightarrow{T}(p) \rangle = 1$ for $\mathcal{H}^k$-a.e. point p in the support of T.

Remark 2.1.4. In the smooth setting, a submanifold Σ is calibrated by α if and only if $\alpha\big|_\Sigma = \mathrm{vol}_\Sigma$.

One of the main reasons behind our interest in calibrated currents (calibrated submanifolds) is that, if they have finite mass (volume), they are homologically-volume minimizing.

Lemma 2.1.5 (Harvey–Lawson [37]). *Let (M, g) be a Remannian manifold and let α be a k-dimensional calibration on it. If T has finite mass and it is calibrated by α, then T is homologically volume-minimizing, i.e. $M(T) \leq M(T')$ for every T' homologous to T (i.e. such that $T - T' = \partial S$, for some S compactly supported). Equality holds if and only if T' is α-calibrated.*

Proof. Let T' and S as in the statement. Then,

$$M(T) = T(\alpha) = (T' + \partial S)(\alpha) = T'(\alpha) + S(d\alpha) = T'(\alpha) \leq M(T'),$$

where we used the definition of calibration and calibrated current in the first equality and in the last inequality. $\square$

More generally, we deduce that calibrated currents (calibrated submanifolds), non-necessarily with finite mass (volume), are locally volume minimizing.

Lemma 2.1.6. *Let (M, g) be a Remannian manifold and let α be a k-dimensional calibration on it. If T is calibrated by α, then it is locally volume minimizing.*

Finding examples of calibration is fairly easy, note, for instance, that $\alpha \equiv 0$ is a calibration on any Riemannian manifold. However, it is far less trivial to find calibrations admitting calibrated submanifolds, or, at least, a "large enough" space of calibrated planes at each point of M. Interesting examples of such calibrations can be found in manifolds of special holonomy (cfr. Section 1.1.2)

Example 2.1.7. These are the examples of interesting calibrations arising in manifolds of special holonomy.

- Let (M, ω) be a Kähler manifold. Then, $\omega^k/k!$ is a calibration for every k and the calibrated submanifolds are the k-dimensional complex submanifolds.
- Let (M, ω, Ω) be a Calabi–Yau manifold. Then, $\mathrm{Re}(e^{i\theta}\Omega)$ is a calibration for every θ and the calibrated submanifolds are called special Lagrangians of phase θ.
- Let (M, φ) be a G_2 manifold. Then, φ and $*\varphi$ are calibrations and the calibrated submanifolds are called associatives and coassociatives, respectively (cfr. Section 2.2).
- Let (M, Φ) be a Spin(7) manifold. Then, Φ is a calibration and the calibrated submanifolds are called Cayleys (cfr. Section 2.3).

2.2 Holonomy G_2

In this section, we recall some basic results concerning G_2 manifolds, associative submanifolds and coassociative submanifolds.

2.2.1 G_2 manifolds

The linear model we consider for a G_2 manifold is $\mathbb{R}^7 \cong \mathbb{R}^3 \oplus \mathbb{R}^4$ parametrized by (x_1, x_2, x_3) and (a_0, a_1, a_2, a_3), respectively. On $\mathbb{R}^7$, we consider the associative 3-form φ_0:

$$\varphi_0 = dx_1 \wedge dx_2 \wedge dx_3 + \sum_{i=1}^{3} dx_i \wedge \Omega_i,$$

where the Ω_is are the standard ASD two-forms of $\mathbb{R}^4$ endowed with the Euclidean metric, i.e., $\Omega_i = da_0 \wedge da_i - da_j \wedge da_k$ for (i, j, k) cyclic permuation of $(1, 2, 3)$. The Hodge dual of φ_0 in $\mathbb{R}^7$ is also of great geometrical interest:

$$*\varphi_0 = da_0 \wedge da_1 \wedge da_2 \wedge da_3 - \sum_{i=1}^{3} dx_j \wedge dx_k \wedge \Omega_i,$$

where (i, j, k) is again a positive permutation of $(1, 2, 3)$.

Since the stabilizer of φ_0 is isomorphic to G_2, the automorphism group of $\mathbb{O}$, we can see $(\mathbb{R}^7, \varphi_0)$ as the linear model for manifolds with G_2-structure group.

Definition 2.2.1. Let M be a manifold and φ a 3-form on M. We say that φ is a G_2-structure on M if at each point $x \in M$ there exists a linear isomorphism $p_x : \mathbb{R}^7 \to T_xM$ which identifies φ_0 with $\varphi\big|_x$, i.e., $p_x^*\varphi = \varphi_0$.

A G_2 structure φ induces a metric g_φ and an orientation vol_φ on M satisfying:

$$(i_u \circ \varphi) \wedge (i_v \circ \varphi) \wedge \varphi = -6 g_\varphi(u, v)\, \mathrm{vol}_\varphi, \tag{2.2.1}$$

for all $u, v \in T_xM$ and all $x \in M$. This makes p_x an orientation preserving isometry. From g_φ and vol_φ, one can also construct the coassociative 4-form $*_\varphi\varphi$. We remand the reader to [66] for further details.

Definition 2.2.2. Let M be a manifold and let φ be a G_2-structure on M. (M, φ) is a G_2 manifold if the G_2-structure is torsion-free, i.e., φ and $*_\varphi\varphi$ are closed (or, equivalently, if φ is closed and co-closed).

Note that Fernandez and Gray showed in [29] that φ is closed and co-closed if and only if φ is parallel. Hence, by Proposition 1.1.4, (M, g_φ) is a G_2 manifold if and only if $\mathrm{Hol}(g_\varphi) \subseteq \mathrm{G}_2$.

Proposition 2.2.3 (Bonan [15], Alekseevsky [4]). *Every G_2 manifold is Ricci-flat.*

The octonionic nature of the tangent space equips the tangent bundle with a natural cross product.

Definition 2.2.4. Let (M, φ) be a manifold with a G_2-structure. The cross product on the tangent bundle $\times_\varphi$ is defined as follows:

$$\begin{aligned} \times_\varphi : & TM \times TM \to TM \\ & (U, V) \to (i_V \circ i_U \varphi)^\#, \end{aligned}$$

where $\#$ denotes the Riemannian musical isomorphism.

In particular, if U, V, W are vector fields on M, then, $U \times_\varphi V$ is characterized by the following equation:

$$\varphi(U, V, W) = g_\varphi(U \times_\varphi V, W).$$

2.2.2 Associative and coassociative submanifolds

Given a manifold with a G_2-structure, (M, φ), it is clear that φ and $*\varphi$ have co-mass equal to one. It follows that, if (M, φ) is a G_2 manifold, then, φ and $*\varphi$ are calibrations.

Definition 2.2.5. Let $F \subset (\mathbb{R}^7, \varphi_0)$ be a 3-dimensional vector subspace. F is an associative plane if $\varphi_0\big|_F = \mathrm{vol}_F$. A submanifold L of a G_2 manifold (M, φ) is associative if it is calibrated by φ, i.e. $\varphi\big|_L = \mathrm{vol}_L$.

Definition 2.2.6. Let $F \subset (\mathbb{R}^7, \varphi_0)$ be a 4-dimensional vector subspace. F is a coassociative plane if $*\varphi_0\big|_F = \mathrm{vol}_F$. A submanifold Σ of a G_2 manifold (M, φ) is coassociative if it is calibrated by $*\varphi$, i.e. $*\varphi\big|_\Sigma = \mathrm{vol}_\Sigma$.

Remark 2.2.7. Obviously, Σ is associative or coassociative if and only if $T_p\Sigma$ is an associative or a coassociative plane of $(\mathbb{R}^7, \varphi_0)$ for every $x \in \Sigma$ under the isomorphism p_x.

We now state some well-known properties of associative and coassociative planes which will be useful in the discussion below.

Proposition 2.2.8 (Harvey–Lawson [37])**.** *Let $F \subset (\mathbb{R}^7, \varphi_0)$ be a 3-dimensional subspace. Then, the following are equivalent:*

1. *F is an associative plane,*
2. *$F^\perp$ is a coassociative plane,*
3. *if $u, v \in F$, then, $u \times_{\varphi_0} v \in F$,*
4. *if $u \in F$ and $v \in F^\perp$, then, $u \times_{\varphi_0} v \in F^\perp$,*

5. *if* $u, v \in F^{\perp}$, *then,* $u \times_{\varphi_0} v \in F$,

6. *if* $u, v, w \in F$, *then,* $i_w \circ i_v \circ i_u *_{\varphi_0} \varphi_0 = 0$

7. *if* $u, v, w \in F^{\perp}$, *then,* $i_w \circ i_v \circ i_u \varphi_0 = 0$.

Moreover, it follows that for every u, v *linearly independent vectors of* $\mathbb{R}^7$ *there exists a unique associative plane containing them. Analogously, if* u, v, w *are linearly independent vectors of* $\mathbb{R}^7$ *such that* $\varphi_0(u, v, w) = 0$ *there exists a unique coassociative plane containing them.*

It is clear that we can translate this statement to the tangent space $(T_xM, \varphi\big|_x)$ of a G_2 manifold through p_x. Moreover, one can also obtain the following local existence and uniqueness theorem.

Theorem 2.2.9 (Local existence and uniqueness; Harvey–Lawson [37])**.** *Let* N *be a real analytic submanifold of a* G_2 *manifold* (M, φ)*. If* N *is* 2*-dimensional, then, there exists a unique associative real-analytic submanifold* L *such that* $N \subset L$*. If* N *is* 3*-dimensional and* $\varphi\big|_N \equiv 0$*, then there exists a unique coassociative real-analytic submanifold* Σ *such that* $N \subset \Sigma$*.*

2.2.3 Blow-up of associatives and coassociatives with symmetries

In this subsection, we recall some preliminary results that we will use to study the singularities of associative and coassociative submanifolds with symmetries.

The first result, due to Madsen and Swann, claims that the blow-up of any torsion-free G_2-structure converges to the standard local model.

Theorem 2.2.10 (Madsen–Swann [64])**.** *Let* φ_0 *be the standard* G_2*-structure of* $\mathbb{R}^7$ *and let* φ *be a torsion-free* G_2*-structure on* $B_2(0) \subset \mathbb{R}^7$ *such that* $\varphi(0) = \varphi_0(0)$*. Then, for* $t > 0$*, the rescaled* G_2*-structure* $\varphi_t := t^{-3}\lambda_t^*\varphi$ *is such that* $\varphi_1 = \varphi$ *and we have that* $\varphi_t \to \varphi_0$ *as* $t \to 0$ *on* $B_1(0)$ *in the* C^k*-norm for every* $k \geq 0$*, where* $\lambda_t(x) := tx$ *for every* $x \in \mathbb{R}^7$*. Moreover, the same holds for the* φ_t*-induced Riemannian metric* $g_t = t^{-2}\lambda_t^*g$ *and dual form* $(*\varphi)_t = t^{-4}\lambda_t^*(*\varphi)$*, where* g *is the Riemannian metric induced by* φ *and* $*$ *is the relative hodge dual.*

Moreover, Harvey and Lawson showed that under the blow-up procedure calibrated currents remain calibrated, and converge to a calibrated tangent cone.

Theorem 2.2.11 (Harvey–Lawson [37]). *If L is a φ-calibrated current in $(B_2(0), \varphi)$, then, $L_t := (\lambda_{t^{-1}})_\sharp L (= t^{-1}L)$ is a φ_t-calibrated current in $(B_2(0), \varphi_t)$ for every $t \in (0, 1]$. Moreover, if $0 \in \mathrm{supp}(L)$, then, L_t converges in the sense of currents, up to subsequences, to a φ_0-calibrated non-empty tangent cone C. The same result holds for $*\varphi$-calibrated currents.*

Proof. Let $p \in \mathrm{supp}(L_t)$ and let X_1, X_2, X_3 be an oriented orthonormal basis of the approximate tangent space of L_t at p. Then, $tp \in \mathrm{supp}(L)$ and X_1, X_2, X_3 is an oriented orthonormal basis of L at tp, where we identified the approximate tangent spaces of L and L_t as vector subspaces of $\mathbb{R}^7$. Hence, the first part follows from the definition of φ_t. The remaining is a consequence of the theory of tangent cones for area-minimizing currents (see [69, Section 7.35]). □

A result due to Simon [68, Corollary p. 564], together with Allard's regularity theorem (see [69, Chapter 5]), allows us to study the geometry of calibrated currents with mild singularities.

Theorem 2.2.12. *If L is a φ-calibrated current in $(B_2(0), \varphi)$ of density 1 away from 0 and has a tangent cone C at 0 that is non-singular (i.e. $C \setminus \{0\}$ is smooth), then, C is the unique tangent cone and, in a smaller neighbourhood of 0, L is smooth everywhere apart from 0, where the singularity is modeled on C. Moreover, if C is also flat, then L is smooth at 0. The same result holds for $*\varphi$-calibrated currents.*

Finally, since we will be interested in G-invariant submanifolds, for some compact Lie group G acting effectively on M, we study how vector fields behave under blow-up. These vector fields will be chosen to be the generators of the action.

Proposition 2.2.13. *Let X be a vector field on $(B_2(0), \varphi)$ such that $\mathcal{L}_X\varphi = 0$. Then, the rescaled vector field $X^t := \lambda_t^* X = t^{-1}(X \circ \lambda_t)$ is such that $\mathcal{L}_{X^t}\varphi_t = 0$. Moreover, the same holds for $f(t)X^t$, where $f \in C^\infty(\mathbb{R}^+; \mathbb{R})$.*

Proof. It follows from a straightforward application of Cartan's formula and $\lambda_t^*(i_X\varphi) = i_{\lambda_t^* X}\lambda_t^*\varphi$. □

Since $[X^t, Y^t] = \lambda_t^*[X, Y]$ for every X, Y vector fields, the generators of a G-action defined for $t = 1$ will give vector fields satisfying the same equations for every $t > 0$. Unfortunately, if we let t go to 0, X^t does not necessarily converge. Indeed, if we write

$$X(x) = \sum_{i=1}^{7} a_i(x)\partial_i,$$

for some functions a_i on $B_2(0)$, then,

$$X^t(x) = t^{-1}\sum_{i=1}^{7} a_i(tx)\partial_i,$$

which does not converge if some $a_i(0) \neq 0$. However, assuming that X is real-analytic, we can always find a minimal integer $\alpha \leq 1$ such that $\tilde{X}^t := t^\alpha X^t$ converges smoothly to some non-zero vector field $\tilde{X}$ as $t \to 0$. Clearly, $\alpha = 1$ if and only if $X(0) \neq 0$. Moreover, if $\mathcal{L}_{X^t}\varphi_t = 0$, then Proposition 2.2.13 implies $0 = \mathcal{L}_{\tilde{X}^t}\varphi_t \to \mathcal{L}_{\tilde{X}}\varphi_0$.

In a similar fashion, given a 1-form ω one can define $\omega_t, \tilde{\omega}_t$ and $\tilde{\omega}$.

Lemma 2.2.14. *Given three vector fields X, Y, Z on $(B_2(0), \varphi)$ as in Theorem 2.2.10, then, for $t \to 0$ the following equations hold:*

1. $(\widetilde{i_X \circ i_Y \varphi})_t = i_{\tilde{X}^t} \circ i_{\tilde{Y}^t}\varphi_t \to i_{\tilde{X}} \circ i_{\tilde{Y}}\varphi_0,$
2. $(\widetilde{i_X \circ i_Y \circ i_Z * \varphi})_t = i_{\tilde{X}^t} \circ i_{\tilde{Y}^t} \circ i_{\tilde{Z}^t} * \varphi_t \to i_{\tilde{X}} \circ i_{\tilde{Y}} \circ i_{\tilde{Z}} * \varphi_0.$

The following lemma shows that if X is a Killing vector field one can choose coordinates in which α is either 0 or 1.

Lemma 2.2.15. *Let $X_1, \dots X_k$ be Killing vector fields on (M, φ) generated by an automorphic group action G, such that $X_1, \dots, X_l$ vanish at p and $X_{l+1}, \dots, X_k$ do not vanish at p. Then, we can choose normal coordinates around p, such that:*

$$\tilde{X}_i = \tilde{X}_i^t = X_i \text{ if } i \leq l,$$
$$\tilde{X}_i = X_i(0) \neq 0 \text{ if } i \geq l+1$$

and $\varphi(0) = \varphi_0$. In particular, this means that the α_i relative to $\tilde{X}^t$ is zero in the first case and one in the second.

Proof. When $i \geq l+1$, the statement holds in any coordinates and is a direct consequence of X_i being continuous.

Normal coordinates are defined via the exponential map $\exp_p : B_\epsilon(0) \subset T_pM \to \mathcal{U} \subset M$. Because of the slice theorem, this map is G-equivariant and the stabilizer group G_p, has Lie algebra which is generated by $X_1, \dots, X_l$. So, in normal coordinates, the vector fields $X_1, \dots, X_l$ generate a linear action on T_pM. This means they agree with their first order approximation and the statement follows. We can use the freedom to choose a basis of T_pM such that $\varphi(0) = \varphi_0$ since $\mathrm{GL}(7, \mathbb{R})$ acts transitively on positive 3-forms on $\mathbb{R}^7$. ☐

We will be interested in the case where the group G is $\mathbb{T}^2 \times \mathrm{SU}(2)$, or some discrete quotient of it. If U_1, U_2 are the generators of the $\mathbb{T}^2$-component and V_1, V_2, V_3 are generators of the $\mathrm{SU}(2)$-component, then, for every $l, m = 1, 2$ and all (i, j, k) cyclic permutation of $(1, 2, 3)$, they satisfy:

$$[U_1, U_2] = 0, \quad [U_l, V_m] = 0, \quad [V_i, V_j] = \epsilon_{ijk} V_k.$$

It follows that the vector fields $\tilde{U}_1^t, \tilde{U}_2^t, \tilde{V}_1^t, \tilde{V}_2^t, \tilde{V}_3^t$ are such that:

$$[\tilde{U}_1^t, \tilde{U}_2^t] = 0, \quad [\tilde{U}_l^t, \tilde{V}_m^t] = 0, \tag{2.2.2}$$

$$[\tilde{V}_i^t, \tilde{V}_j^t] = t^{\alpha_i + \alpha_j - \alpha_k} \tilde{V}_k^t, \tag{2.2.3}$$

where α_i is the α defining $\tilde{V}_i^t$.

2.2.4 The Bryant–Salamon G_2 manifold of topology $S^3 \times \mathbb{R}^4$

In this section we describe the Bryant–Salamon G_2 manifold of topology $S^3 \times \mathbb{R}^4$ [19]. In their work, Bryant and Salamon constructed a 1-parameter family of torsion-free G_2-structures on the spinor bundle over S^3, which is a trivial bundle in this case. The 3-dimensional sphere is endowed with the metric of constant sectional curvature k, which we can assume to be equal to one up to rescalings. Further details can be found in [19] and [49].

Remark 2.2.16. The construction that we describe on S^3 works on manifolds and orbifolds with negative constant sectional curvature. However, in these cases, the metric is not complete or smooth.

2.2.4.1 The spinor bundle over S^3

Let S^3 be the 3-sphere endowed with the Riemannian metric of constant curvature 1. Given an oriented orthonormal frame of S^3, $\{e_1, e_2, e_3\}$, we can construct the dual oriented orthonormal coframe, $\{b_1, b_2, b_3\}$, and the relative Levi-Civita connection matrix:

$$\rho = \begin{pmatrix} 0 & -2\rho_3 & 2\rho_2 \\ 2\rho_3 & 0 & -2\rho_1 \\ -2\rho_2 & 2\rho_1 & 0 \end{pmatrix},$$

which is determined by the first structure equation: $d\underline{b} = -\rho \wedge \underline{b}$. In particular, we will consider e_1, e_2, e_3 to be the left-invariant frame of $S^3 \cong \mathrm{Sp}(1)$ such that $[e_i, e_j] = -2e_k$

and, hence, $db_i = 2b_j \wedge b_k$ for (i,j,k) positive permutation of $(1,2,3)$. The metric and the volume form in this frame become:

$$g_{S^3} = b_1^2 + b_2^2 + b_3^2, \qquad \mathrm{vol}_{S^3} = b_1 \wedge b_2 \wedge b_3,$$

while the Levi-Civita connection matrix is determined by: $\rho_i = -\frac{1}{2} b_i$.

Now, consider $S^3 \times \mathbb{R}^4$ as a trivial vector bundle over S^3, and endow it with the connection induced by the matrix:

$$\rho = \begin{pmatrix} 0 & \rho_1 & \rho_2 & \rho_3 \\ -\rho_1 & 0 & -\rho_3 & \rho_2 \\ -\rho_2 & \rho_3 & 0 & -\rho_1 \\ -\rho_3 & -\rho_2 & \rho_1 & 0 \end{pmatrix}.$$

Remark 2.2.17. Equivalently, ρ is the spin connection on the spinor bundle $\not{S}(S^3)$.

The vertical one forms with respect to this connection are:

$$\xi_0 = da_0 + a_1\rho_1 + a_2\rho_2 + a_3\rho_3, \qquad \xi_1 = da_1 - a_0\rho_1 + a_3\rho_2 - a_2\rho_3,$$
$$\xi_2 = da_2 - a_3\rho_1 - a_0\rho_2 + a_1\rho_3, \qquad \xi_3 = da_3 + a_2\rho_1 - a_1\rho_2 - a_0\rho_3,$$

where (a_0, a_1, a_2, a_3) are the coordinates on the fibers. Recall that the horizontal 1-forms are spanned by $\{\pi^*_{S^3}(b_i)\}$, where $\pi_{S^3} : \not{S}(S^3) \to S^3$ is the usual bundle projection. As an abuse of notation, we will omit the pullback symbol.

2.2.4.2 The G_2-structure

Now that we recalled the geometry of the vertical and horizontal spaces, we are ready to define the Bryant–Salamon construction of torsion-free G_2-structures on $S^3 \times \mathbb{R}^4$. Let $r^2 := a_0^2 + a_1^2 + a_2^2 + a_3^2$, which corresponds to the square of the distance from the zero section, and let

$$\Omega_1 = \xi_0 \wedge \xi_1 - \xi_2 \wedge \xi_3,\ \Omega_2 = \xi_0 \wedge \xi_2 - \xi_3 \wedge \xi_1,\ \Omega_3 = \xi_0 \wedge \xi_3 - \xi_1 \wedge \xi_2,$$

then, the G_2-structures on $\not{S}(S^3)$ given by Bryant and Salamon are:

$$\varphi_c = f^3 \mathrm{vol}_{S^3} + f g^2 \sum_{i=1}^{3} b_i \wedge \Omega_i,$$

where $f = \sqrt{3}(c+r^2)^{\frac{1}{3}}$ and $g = 2(c+r^2)^{-1/6}$. The induced metric, the relative coassociative and volume forms become:

$$g_c = f^2 g_{S^3} + g^2(\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2),$$
$$*_{\varphi_c}\varphi_c = g^4 \xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 - f^2 g^2 \sum_{i=1}^{3} b_j \wedge b_k \wedge \Omega_i,$$
$$\mathrm{vol}_c = f^3 g^4 \mathrm{vol}_{S^3} \wedge \xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3.$$

As usual, (i,j,k) is a positive permutation of $(1,2,3)$.

Setting $c=0$ and $M_0 := \not{S}(S^3) \setminus S^3$, we obtain a G_2 cone (M_0,φ_0), i.e. M_0 with the metric induced by the G_2-structure φ_0 is a Riemannian cone.

Theorem 2.2.18 (Bryant–Salamon [19])**.** *Let (M_c,φ_c) be the spinor bundle of S^3 (or the relative cone) endowed with the Bryant-Salamon G_2 structure φ_c, $c\geq 0$. Then, $d\varphi_c=0$, $d*_{\varphi_c}\varphi_c=0$ and $\mathrm{Hol}(M_c,g_c)=\mathrm{G}_2$.*

Moreover, $\mathrm{SU}(2)^3\cong \mathrm{Sp}(1)^3$ acting on $S^3\times\mathbb{R}^4\cong S^3\times\mathbb{H}\subset\mathbb{H}^2$ as follows:

$$(q_1,q_2,q_3)\cdot(x,a)=(q_1x\bar{q_3},q_2a\bar{q_3}),$$

for $(q_1,q_2,q_3)\in\mathrm{SU}(2)^3$ and $(x,a)\in S^3\times\mathbb{H}$, is structure preserving.

Remark 2.2.19. The functions f and g defined above satisfy the following equations:

$$\dot{(f^3)}=\frac{3k}{4}fg^2,\quad \dot{(fg^2)}=0,\quad \dot{(f^2g^2)}=\frac{k}{4}g^4, \tag{2.2.4}$$

for $k=1$ and where the dot represents the derivative with respect to r^2.

In general, the Bryant–Salamon torsion-free G_2-structures on the spinor bundle over a 3-manifold of constant sectional curvature k are characterized by the forms:

$$\varphi := f^3 b_1\wedge b_2\wedge b_3 + fg^2\sum_{i=1}^{3} b_i\wedge(\xi_0\wedge\xi_i-\xi_j\wedge\xi_k),$$
$$*\varphi := g^4\xi_0\wedge\xi_1\wedge\xi_2\wedge\xi_3 - f^2g^2\sum_{i=1}^{3} b_j\wedge b_k\wedge(\xi_0\wedge\xi_i-\xi_j\wedge\xi_k),$$

with f,g satisfying Eq. (2.2.4).

2.2.5 The Foscolo–Haskins–Nordström manifolds

In this section, we provide a brief description of the G_2 manifolds constructed by Foscolo, Haskins and Nordström in [32], which we will refer to FHN manifolds for brevity.

2.2.5.1 The topology of the FHN manifolds

Let (M,φ) be a non-compact, simply-connected G_2 manifold, with a structure-preserving $\mathrm{SU}(2)\times\mathrm{SU}(2)$ cohomogeneity-one action. Then, it is well-known that $M/\mathrm{SU}(2)\times\mathrm{SU}(2)$ is an open or half-closed interval I, and hence, the cohomogeneity-one structure can be encoded by a pair of closed subgroups: $K_0\subset K\subset\mathrm{SU}(2)\times\mathrm{SU}(2)$, which are referred to as the group diagram of M. In particular, $\mathrm{SU}(2)\times\mathrm{SU}(2)/K_0$ is diffeomorphic to the principal orbits of the $\mathrm{SU}(2)\times\mathrm{SU}(2)$-action and corresponds to the interior of I, while

$\mathrm{SU}(2) \times \mathrm{SU}(2)/K$ is diffeomorphic to the singular orbit and corresponds to the boundary of I, if it exists.

In the case of our interest, we either have $K_0 = \{1_{\mathrm{SU}(2)\times\mathrm{SU}(2)}\}$ or $K_0 = K_{m,n} \cap K_{2,-2}$, where m, n are coprime integers and $K_{m,n} \cong U(1) \times \mathbb{Z}_{\gcd(n,m)}$ is defined by:

$$K_{m,n} := \left\{(e^{i\theta_1}, e^{i\theta_2}) \in \mathbb{T}^2 : e^{i(m\theta_1+n\theta_2)} = 1\right\} \subset \mathrm{SU}(2) \times \mathrm{SU}(2),$$

where $\mathbb{T}^2$ is the maximal torus in $\mathrm{SU}(2) \times \mathrm{SU}(2)$. If m, n are coprime the isomorphism between $K_{m,n} \subset \mathrm{SU}(2) \times \mathrm{SU}(2)$ and $U(1)$ is:

$$e^{i\theta} \mapsto (e^{in\theta}, e^{-im\theta}), \tag{2.2.5}$$

moreover, $K_{m,n} \cap K_{2,-2} \cong \mathbb{Z}_{2|m+n|}$. Up to automorphisms of $\mathrm{SU}(2) \times \mathrm{SU}(2)$, the subgroup K determining the singular orbit $\mathrm{SU}(2) \times \mathrm{SU}(2)/K$ is one of the following:

$$\Delta\,\mathrm{SU}(2), \quad \left\{1_{\mathrm{SU}(2)}\right\} \times \mathrm{SU}(2), \quad K_{m,n},$$

where $\Delta\,\mathrm{SU}(2)$ denotes the $\mathrm{SU}(2)$ sitting diagonally in $\mathrm{SU}(2) \times \mathrm{SU}(2)$. Note that the singular orbit is diffeomorphic to S^3 for the first two cases, and to $S^2 \times S^3$ for the third one.

2.2.5.2 The G_2-structure

We now describe the G_2-structure on the principal part of M, diffeomorphic to $(\mathrm{SU}(2) \times \mathrm{SU}(2))/K_0 \times \mathrm{Int}(I)$.

Consider on $\mathrm{SU}(2) \times \mathrm{SU}(2)$ the basis $\{b_1, b_2, b_3, \tilde{b}_1, \tilde{b}_2, \tilde{b}_3\}$ of left-invariant 1-forms satisfying:

$$db_i = 2b_j \wedge b_k, \quad d\tilde{b}_i = 2\tilde{b}_j \wedge \tilde{b}_k,$$

and denote by $e_1, e_2, e_3, f_1, f_2, f_3$ the dual vector fields. On the principal part of M, these can be explicitly described as follows:

$$\begin{aligned} e_1(p,q,r) &= -(pi,0,0), \quad & e_2(p,q,r) &= -(pj,0,0), \quad & e_3(p,q,r) &= -(pk,0,0),\\ f_1(p,q,r) &= -(0,qi,0), \quad & f_2(p,q,r) &= -(0,qj,0), \quad & f_3(p,q,r) &= -(0,qk,0), \end{aligned}$$

where the product is by quaternionic multiplication. Let $c_1, c_2 \in \mathbb{R}$ and let a_1, a_2, a_3 be three functions only depending on the interval I. The following closed 3-form on $(\mathrm{SU}(2) \times \mathrm{SU}(2))/K_0 \times \mathrm{Int}(I)$:

$$\varphi = -8c_1 b_1 \wedge b_2 \wedge b_3 - 8c_2 \tilde{b}_1 \wedge \tilde{b}_2 \wedge \tilde{b}_3 + 4d(a_1 b_1 \wedge \tilde{b}_1 + a_2 b_2 \wedge \tilde{b}_2 + a_3 b_3 \wedge \tilde{b}_3) \tag{2.2.6}$$

is a G_2-structure if and only if the following conditions are satisfied:

$$\dot{a}_i > 0, \quad \Lambda(a_1, a_2, a_3) < 0, \quad 2\dot{a}_1\dot{a}_2\dot{a}_3 = \sqrt{-\Lambda(a_1, a_2, a_3)},$$

where

$$\begin{aligned}\Lambda(a_1, a_2, a_3) =& a_1^4 + a_2^4 + a_3^4 - 2a_1^2a_2^2 - 2a_2^2a_3^2 - 2a_3^2a_1^2 + 4(c_1 - c_2)a_1a_2a_3 + \\ &+ 2c_1c_2(a_1^2 + a_2^2 + a_3^2) + c_1^2c_2^2.\end{aligned}$$

Furthermore, if $K_0 = K_{m,n} \cap K_{2,-2}$, we require $a_2 = a_3$ unless there exists a $d \in \mathbb{Z}$ such that $(d+1)m + (d-1)n = 0$.

Remark 2.2.20. Under these conditions, the interval I is the arc-length parameter along a geodesic meeting all the principal orbits orthogonally.

The torsion-free condition becomes the Hamiltonian system associated to the potential:

$$H(x, y) = \sqrt{-\Lambda(y_1, y_2, y_3)} - 2\sqrt{x_1x_2x_3},$$

where $y_i = a_i$ and $x_i = \dot{a}_j\dot{a}_k$ for every (i, j, k) cyclic permutation of $(1, 2, 3)$. If t denotes the parametrization of I, then, the dual form of φ is given by:

$$\begin{aligned}*\varphi =& 16\sum_{i=1}^{3} \dot{a}_j\dot{a}_k b_j \wedge \tilde{b}_j \wedge b_k \wedge \tilde{b}_k + \\ &+ \frac{8}{\sqrt{-\Lambda}} dt \wedge \Bigg((2a_1a_2a_3 - c_1(a_1^2 + a_2^2 + a_3^2 + c_1c_2))b_1 \wedge b_2 \wedge b_3 \\ &\qquad + (2a_1a_2a_3 + c_2(a_1^2 + a_2^2 + a_3^2 + c_1c_2))\tilde{b}_1 \wedge \tilde{b}_2 \wedge \tilde{b}_3 \\ &\qquad + \sum_{i=1}^{3} \big((a_i(a_i^2 - a_j^2 - a_k^2 + c_1c_2) - 2c_2a_ja_k)b_i \wedge \tilde{b}_j \wedge \tilde{b}_k \\ &\qquad\qquad + (a_i(a_i^2 - a_j^2 - a_k^2 + c_1c_2) + 2c_1a_ja_k)\tilde{b}_i \wedge b_j \wedge b_k\big)\Bigg).\end{aligned} \tag{2.2.7}$$

Enhanced symmetry. We now restrict our discussion to the case where $a := a_2 = a_3$ and $b := a_1$. Under this additional condition, the symmetry of $(\mathrm{SU}(2)\times\mathrm{SU}(2))/K_0\times\mathrm{Int}(I)$ becomes $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$, where the action of $(\gamma_1, \gamma_2, \lambda) \in \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ on $([p, q], t) \in (\mathrm{SU}(2) \times \mathrm{SU}(2))/K_0 \times \mathrm{Int}(I)$ is as follows:

$$(\gamma_1, \gamma_2, \lambda) \cdot ([p, q], t) = ([\gamma_1 p\overline{\lambda}, \gamma_2 q\overline{\lambda}], t), \tag{2.2.8}$$

where λ is given by the $\mathrm{U}(1) \subset \mathrm{SU}(2)$ generated by quaternionic multiplication by i.

Under this enhanced symmetry, the form of $\Lambda(a, b)$ simplifies to:

$$-\Lambda(a, b) = 4a^2(b - c_1)(b + c_2) - (b^2 + c_1c_2)^2,$$

and the same holds for the Hamiltonian system, which becomes:

$$\dot{x}_1 = -\frac{\Lambda_a(y_1, y_2)}{4\sqrt{-\Lambda(y_1, y_2)}}, \quad \dot{x}_2 = -\frac{\Lambda_b(y_1, y_2)}{2\sqrt{-\Lambda(y_1, y_2)}},$$
$$\dot{y}_1 = \frac{x_1 x_2}{\sqrt{x_1^2 x_2}}, \qquad \dot{y}_2 = \frac{x_1^2}{\sqrt{x_1^2 x_2}},$$

where $y_1 = a, y_2 = b, x_1 = \dot{a}\dot{b}, x_2 = \dot{a}^2$ and Λ_a, Λ_b denote the derivative of $\Lambda(a, b)$ with respect to the first or the second component, respectively.

Remark 2.2.21. From $-\Lambda(a, b) > 0$, we deduce that $a, b - c_1, b + c_2$ have definite sign, and hence, $\dot{x}_1$ has definite sign as well.

Example 2.2.22. The Bryant–Salamon manifolds described in Section 2.2.4 can be seen as special examples of FHN manifolds such that, for some $c > 0$:

$$a_1 = a_2 = a_3 = \frac{\sqrt{3}}{2}r^2, \quad c_1 = -\frac{3}{8}\sqrt{3}c, \quad c_2 = 0, \quad K = \left\{1_{\mathrm{SU}(2)}\right\} \times \mathrm{SU}(2) \tag{2.2.9}$$

or

$$a_1 = a_2 = a_3 = \frac{1}{6}r^3 - \frac{1}{3}c^3, \quad c_1 = -c_2 = c^3, \quad K = \Delta\,\mathrm{SU}(2),$$

where $r(t)$ is a reparametrization of t such that $dr/dt = 1/2(c + r^2)^{1/6}$ in the first case and $dr/dt = 1/\sqrt{3}\sqrt{1 - 8c^3 r^{-3}}$ in the second case.

Example 2.2.23. The G_2 manifolds predicted by Brandhuber–Gomis–Gubser–Gukov in [16] (rigourously constructed by Bogoyavlenskaya in [14]) can also be seen as special examples of FHN manifolds.

2.2.5.3 Extension to the singular orbit and forward completeness

Now, we state under which conditions the G_2-structure extends smoothly to the singular orbit and when it is forward complete.

First, we know from the slice theorem that a neighbourhood of the singular orbit $\mathrm{SU}(2) \times \mathrm{SU}(2)/K$ is equivariantly diffeomorphic to a small disk bundle of:

$$(\mathrm{SU}(2) \times \mathrm{SU}(2)) \times_K V,$$

for some vector space V endowed with a representation of K. We now summarise when the G_2-structure defined in Eq. (2.2.6) extends smoothly to the zero section of such a bundle (cfr. [32, Proposition 4.1]).

Case 1 ($K = \Delta\,\mathrm{SU}(2)$). In this case, $V = \mathbb{C}^2$ and $\mathrm{SU}(2)$ acts in the usual way on it. The $\mathrm{SU}(2) \times \mathrm{SU}(2)$-invariant G_2-structure defined above extends smoothly to the zero-section if and only if:

1. $c_1 + c_2 = 0$,

2. the functions $\{a_i\}$ are even and have the following development near 0: $a_i(t) = c_1 + \frac{1}{2}\alpha t^2 + O(t^4)$ for some $\alpha \in \mathbb{R}$,

3. $8\alpha^3 = c_1 > 0$.

Case 2 ($K = \{1_{\mathrm{SU}(2)}\} \times \mathrm{SU}(2)$). As in the previous case, $V = \mathbb{C}^2$ and SU(2) acts in the usual way on it. The G_2-structure defined above extends smoothly to the zero-section if and only if:

1. $c_2 = 0$,

2. the functions $\{a_i\}$ are even and have the following development near 0: $a_i(t) = \frac{1}{2}\alpha_i t^2 + O(t^4)$ for some $\alpha_i \in \mathbb{R}^+$,

3. $8\alpha_1\alpha_2\alpha_3 = -c_1 > 0$.

Case 3 ($K = K_{m,n}$). In this situation, $V = \mathbb{R}^2$ and $K_{m,n} \cong \mathrm{U}(1)$ acts on it with weight $2|m+n|$. The G_2-structure defined above extends smoothly to the zero-section if and only if:

1. $mn > 0$,

2. $c_1 = -m^2 r_0^3$ and $c_2 = n^2 r_0^3$ for some $r_0 \in \mathbb{R} \setminus \{0\}$,

3. the function a_1 is even and satisfies: $a_1(0) = mnr_0^3$, $\ddot{a}_1(0) > 0$,

4. the function $a_2 + a_3$ is odd and satisfies: $\dot{a}_2(0) + \dot{a}_3(0) > 0$,

5. we either have $a_2 = a_3$ or $m = n = \pm 1$; if the a_2 and a_3 do not coincide, then, their difference is an even function with $|a_2(0) - a_3(0)| < 2|r_0|^3$.

The forward completeness of the local solutions constructed above is discussed in [32, Section 6, Section 7] for the case we have the enhanced symmetry $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. Moreover, they showed that the complete G_2 manifolds they obtain are all the possible complete G_2-manifolds with $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1)$-symmetry.

2.3 Holonomy Spin(7)

In this section, we recall some basic results concerning Spin(7) manifolds and Cayley submanifolds.

2.3.1 Spin(7) manifolds

The local model is $\mathbb{R}^8 \cong \mathbb{R}^4 \oplus \mathbb{R}^4$ with coordinates $(x_0, ..., x_3, a_0, ..., a_3)$, and Cayley form:

$$\Phi_0 = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3 + da_0 \wedge da_1 \wedge da_2 \wedge da_3 + \sum_{i=1}^{3} \omega_i \wedge \eta_i,$$

where $\omega_i = dx_0 \wedge dx_i - dx_j \wedge dx_k$, $\eta_i = da_0 \wedge da_i - da_j \wedge da_k$ and (i, j, k) is a cyclic permutation of $(1, 2, 3)$. Note that $\{\omega_i\}_{i=1}^3$ and $\{\eta_i\}_{i=1}^3$ are the standard basis of the anti-self-dual 2-forms on the two copies of $\mathbb{R}^4$.

It is well-known that Spin(7) is isomorphic to the stabilizer of Φ_0 in $\mathrm{GL}(8, \mathbb{R})$. Hence, we can see $(\mathbb{R}^8, \Phi_0)$ as the linear model for manifolds with Spin(7)-structure group.

Definition 2.3.1. Let M be a manifold and let Φ be a 4-form on M. We say that Φ is a Spin(7)-structure on M if at each point $x \in M$ there exists an oriented isomorphism $p_x : \mathbb{R}^8 \to T_xM$ which identifies Φ_0 with $\Phi\big|_x$, i.e., $p_x^*\Phi = \Phi_0$.

The Spin(7)-structure on M also induces a Riemannian metric, g_Φ, and an orientation, vol_Φ, on M. With respect to these structures Φ is self-dual. We remand the reader to [66] for further details.

Definition 2.3.2. Let M be a manifold and let Φ be a Spin(7)-structure on M. We say that (M, Φ) is a Spin(7) manifold if the Spin(7)-structure is torsion-free, i.e., $d\Phi = 0$.

Similarly to the G_2 case, Bryant [18] showed that Φ is closed if and only if Φ is parallel. Hence, by Proposition 1.1.4, (M, g_Φ) is a Spin(7) manifold if and only if $\mathrm{Hol}(g_\Phi) \subseteq \mathrm{Spin}(7)$.

Proposition 2.3.3 (Alekseevsky [4]). *Every* Spin(7) *manifold is Ricci-flat.*

The octonionic nature of the tangent space equips the tangent bundle with a natural triple cross product.

Definition 2.3.4. Let (M, Φ) be a manifold with a Spin(7)-structure. The triple cross product on the tangent bundle is defined as the musical dual of the following map:

$$\begin{aligned} B :& TM \times TM \times TM \to T^*M \\ & (U, V, W) \to i_W \circ i_V \circ i_U \circ \Phi. \end{aligned}$$

Explicitely, this is given by:

$$U \times V \times W = B(U, V, W)^\#$$

or, equivalently by

$$\Phi(U, V, W, Z) = g_\Phi(U \times V \times W, Z),$$

where $\#$ denotes the Riemannian musical isomorphism and U, V, W, Z are vector fields of M.

2.3.2 Cayley submanifolds

Given a manifold with a Spin(7)-structure, (M, Φ), it is clear that Φ have co-mass equal to one. It follows that, if (M, Φ) is a Spin(7) manifold, then, Φ is a calibration.

Definition 2.3.5. Let $F \subset (\mathbb{R}^8, \Phi_0)$ be a 4-dimensional vector subspace. F is a Cayley plane if $\Phi_0\big|_F = \mathrm{vol}_F$. A submanifold L of a Spin(7) manifold (M, Φ) is Cayley if it is calibrated by Φ, i.e. $\Phi\big|_L = \mathrm{vol}_L$.

Remark 2.3.6. Obviously, L is Cayley if and only if T_pL is a Cayley plane of $(\mathbb{R}^8, \Phi_0)$ for every $x \in L$ under the isomorphism p_x.

We now state some well-known properties of Cayley planes.

Proposition 2.3.7 (Harvey–Lawson [37])**.** *Let $F \subset (\mathbb{R}^8, \Phi_0)$ be a 4-dimensional subspace. Then, the following are equivalent:*

1. *F is a Cayley plane,*
2. *$F^\perp$ is a Cayley plane,*
3. *if $u, v, w \in F$, then, $u \times v \times w \in F$,*
4. *if $u, v \in F$ and $w \in F^\perp$, then, $u \times v \times w \in F^\perp$,*
5. *if $u, v, w \in F^\perp$, then, $u \times v \times w \in F^\perp$,*
6. *if $u, v, w \in F$ and $z \in F^\perp$, then, $i_z \circ i_w \circ i_v \circ i_u \Phi_0 = 0$.*

Moreover, it follows that for every u, v, w linearly independent vectors of $\mathbb{R}^8$ there exists a unique Cayley plane containing them.

It is clear that we can translate this statement to the tangent space $(T_xM, \Phi\big|_x)$ of a Spin(7) manifold through p_x. In particular, one can also obtain the following local existence and uniqueness theorem.

Theorem 2.3.8 (Local existence and uniqueness; Harvey–Lawson [37])**.** *Let N be a 3-dimensional real analytic submanifold of a* Spin(7) *manifold (M, Φ). Then, there exists a unique Cayley real-analytic submanifold Σ such that $N \subset \Sigma$.*

We now give Karigiannis and Min-Oo characterization of the Cayley condition.

Proposition 2.3.9 (Karigiannis–Min-Oo [50, Proposition 2.5]). *The subspace spanned by tangent vectors u, v, w, y is a Cayley 4-plane, up to orientation, if and only if the following form vanishes:*

$$\eta = \pi_7 \left(u^\flat \wedge B(v,w,y) + v^\flat \wedge B(w,u,y) + w^\flat \wedge B(u,v,y) + y^\flat \wedge B(v,u,w) \right),$$

where

$$\pi_7(u^\flat \wedge v^\flat) := \frac{1}{4} \left(u^\flat \wedge v^\flat + i_u \circ i_v \circ \Phi \right).$$

Remark 2.3.10. The reduction of the structure group of M to Spin(7) induces an orthogonal decomposition of the space of differential k-forms for every k, which corresponds to an irreducible representation of Spin(7). In particular, if $k = 2$, the irreducible representations of Spin(7) are of dimension 7 and 21. At each point $x \in M$, these representations induce the decomposition of $\Lambda^2(T_x^*M)$ into two subspaces, which we denote by Λ^2_7 and Λ^2_{21}, respectively. The map π_7 defined in Proposition 2.3.9 is precisely the projection map from the space of two-forms to Λ^2_7. Further details can be found in [66].

2.3.3 The Bryant–Salamon Spin(7)-manifold

In this section we will describe the Spin(7) manifolds constructed by Bryant and Salamon in [19]. There, they described a 1-parameter family of torsion-free Spin(7)-structures on $M := \not{S}_-(S^4)$, the negative spinor bundle on S^4. The 4-dimensional sphere is endowed with the metric of constant sectional curvature k, which is the unique spin self-dual Einstein 4-manifold with positive scalar curvature [40]. Without loss of generality, we can rescale the sphere so that $k = 1$.

Remark 2.3.11. The Bryant–Salamon construction on S^4 also works on spin 4-manifolds with self-dual Einstein metric, but negative scalar curvature, and on spin orbifolds with self-dual Einstein metric. However, in these cases, the metric is not complete or smooth.

2.3.3.1 The negative spinor bundle of S^4

Let S^4 be the 4-sphere endowed with the Riemannian metric of constant sectional curvature 1. As S^4 is clearly spin, given $P_{\mathrm{SO}(4)}$ frame bundle of S^4 we can find the spin structure $P_{\mathrm{Spin}(4)}$ together with the spin representation:

$$\mu := (\mu_+, \mu_-) : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \cong \mathrm{Spin}(4) \to \mathrm{GL}(\mathbb{H}) \times \mathrm{GL}(\mathbb{H}),$$

where $\mu_\pm(p_\pm)(v) := v\overline{p}_\pm$. Let $\tilde{\pi} : P_{\mathrm{Spin}(4)} \to P_{\mathrm{SO}(4)}$ be the double cover in the definition of spin structure, and let $\tilde{\pi}_0^n : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ be the double (universal) covering map for all $n \geq 3$. The negative spinor bundle over S^4 is defined as the associated bundle:

$$\not{S}_-(S^4) := P_{\mathrm{Spin}(4)} \times_{\mu_-} \mathbb{H}.$$

The positive spinor bundle is defined analogously, taking μ_+ instead.

Given an oriented local orthonormal frame for S^4, $\{e_0, e_1, e_2, e_3\}$, the real volume element $e_0 \cdot e_1 \cdot e_2 \cdot e_3$ acts as the identity on the negative spinors and as minus the identity on the positive ones. Now, let $\{b_0, b_1, b_2, b_3\}$ be the dual coframe of $\{e_0, e_1, e_2, e_3\}$, let $\tilde{\omega}$ the connection 1-forms relative to the Levi-Civita connection of S^4 with respect to the frame $\{e_0, e_1, e_2, e_3\}$ and let $\{\sigma_1, \sigma_i, \sigma_j, \sigma_k\}$ a local orthonormal frame for the negative spinor bundle corresponding to the standard basis of $\{1, i, j, k\}$ in this trivialization. Hence, we can define the linear coordinates (a_0, a_1, a_2, a_3) which parametrize a point in the fibre as $a_0\sigma_1 + a_1\sigma_i + a_2\sigma_j + a_3\sigma_k$.

By the properties of the spin connection and the fact we are working on the negative spinor bundle, we can write:

$$\begin{aligned}\nabla\sigma_\alpha &= \left(\rho_1\mu_-(e_2 \cdot e_3) + \rho_2\mu_-(e_3 \cdot e_1) + \rho_3\mu_-(e_1 \cdot e_2)\right)\sigma_\alpha \\ &= \left(\rho_1\mu_-(i) + \rho_2\mu_-(j) + \rho_3\mu_-(k)\right)\sigma_\alpha,\end{aligned}$$

where $2\rho_1 = \tilde{\omega}_2^3 - \tilde{\omega}_0^1$, $2\rho_2 = -\tilde{\omega}_0^2 - \tilde{\omega}_1^3$ and $2\rho_3 = \tilde{\omega}_1^2 - \tilde{\omega}_0^3$. It is well-known that these are the connection forms on the bundle of anti-self-dual 2-forms, with respect to the connection induced by the Levi-Civita connection on S^4 and the frame given by $\Omega_i := b_0 \wedge b_i - b_j \wedge b_k$. As usual, (i, j, k) is a cyclic permutation of $(1, 2, 3)$. The ρ_is are characterized by:

$$d\begin{pmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\end{pmatrix} = -\begin{pmatrix}0 & -2\rho_3 & 2\rho_2\\ 2\rho_3 & 0 & -2\rho_1\\ -2\rho_2 & 2\rho_1 & 0\end{pmatrix} \wedge \begin{pmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\end{pmatrix}, \tag{2.3.1}$$

and the vertical one forms are:

$$\begin{aligned}\xi_0 &= da_0 + \rho_1 a_1 + \rho_2 a_2 + \rho_3 a_3, &\quad \xi_1 &= da_1 - \rho_1 a_0 - \rho_3 a_2 + \rho_2 a_3,\\ \xi_2 &= da_2 - \rho_2 a_0 + \rho_3 a_1 - \rho_1 a_3, &\quad \xi_3 &= da_3 - \rho_3 a_0 - \rho_2 a_1 + \rho_1 a_2.\end{aligned} \tag{2.3.2}$$

Recall that the horizontal 1-forms are spanned by $\{\pi_{S^4}^*(b_i)\}_{i=1}^4$, where π_{S^4} is the usual bundle projection. As above we will omit the pull-back as an abuse of notation.

Remark 2.3.12. A detailed account of spin geometry can be found in [54]. Observe that, there, the definition of positive and negative spinors is interchanged. We opted to stay consistent with [19]. Indeed, the vertical 1-forms we obtain coincide with the ones obtained by Bryant and Salamon, up to renaming the ρ_is.

2.3.3.2 The $\mathrm{Spin}(7)$-structures

If $r^2 := a_0^2 + a_1^2 + a_2^2 + a_3^2$ is the square of the distance function from the zero section and c is a positive constant, then, the $\mathrm{Spin}(7)$-structures defined by Bryant and Salamon are:

$$\begin{aligned}\Phi_c :=&16(c+r^2)^{-4/5}\xi_0\wedge\xi_1\wedge\xi_2\wedge\xi_3 + 25(c+r^2)^{6/5}b_0\wedge b_1\wedge b_2\wedge b_3\\&+20(c+r^2)^{1/5}(A_1\wedge\Omega_1 + A_2\wedge\Omega_2 + A_3\wedge\Omega_3),\end{aligned} \tag{2.3.3}$$

where $A_i := \xi_0\wedge\xi_i - \xi_j\wedge\xi_k$. As usual, (i,j,k) is a cyclic permutation of $(1,2,3)$.

The metric induced by Φ_c is

$$g_c := 4(c+r^2)^{-2/5}(\xi_0^2+\xi_1^2+\xi_2^2+\xi_3^2) + 5(c+r^2)^{3/5}(b_0^2+b_1^2+b_2^2+b_3^2), \tag{2.3.4}$$

while the induced volume element is

$$\mathrm{vol}_c := (20)^2\,(c+r^2)^{2/5}(\xi_0\wedge\xi_1\wedge\xi_2\wedge\xi_3\wedge b_0\wedge b_1\wedge b_2\wedge b_3). \tag{2.3.5}$$

Setting $c=0$ and $M_0 := \not{S}_-(S^4)\setminus S^4 \cong \mathbb{R}^+\times S^7$, we obtain a $\mathrm{Spin}(7)$ cone (M_0,Φ_0), i.e. M_0 with the metric induced by the $\mathrm{Spin}(7)$-structure Φ_0 is a Riemannian cone.

Theorem 2.3.13 (Bryant–Salamon [19]). *Let (M_c,Φ_c) be the spinor bundle of S^4 (or the relative cone) endowed with the Bryant-Salamon $\mathrm{Spin}(7)$-structure Φ_c, $c\geq 0$. Then, $d\Phi_c = 0$ and $\mathrm{Hol}(M_c,g_c) = \mathrm{Spin}(7)$.*

Remark 2.3.14. If we define $f(r^2) := 5(c+r^2)^{3/5}$ and $g(r^2) := 4(c+r^2)^{-2/5}$, then, these functions satisfy the following equations:

$$(\dot{fg}) = \frac{k}{4}g^2, \quad (\dot{f^2}) = \frac{3k}{2}fg, \tag{2.3.6}$$

where the dot denotes the derivative with respect to r^2.

In general, the Bryant–Salamon torsion-free $\mathrm{Spin}(7)$-structures on the negative spinor bundle over a self-dual Einstein 4 manifold of scalar curvature k are characterized by the Cayley 4-form:

$$\Phi := g^2\xi_0\wedge\xi_1\wedge\xi_2\wedge\xi_3 + f^2 b_0\wedge b_1\wedge b_2\wedge b_3 + fg\sum_{i=1}^{3}A_i\wedge\Omega_i,$$

with f,g satisfying Eq. (2.3.6).

2.3.3.3 Automorphism Group

In the setting we are considering, Bryant and Salamon noticed that the diffeomorphisms given by the $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$-action described as follows are actually the automorphism group [19, Theorem 2]. Consider $\mathrm{SO}(5)$ acting on S^4 in the standard way. This induces an action on the frame bundle of S^4 via the differential, which easily lifts to a $\mathrm{Spin}(5) \cong \mathrm{Sp}(2)$ action on $P_{\mathrm{Spin}(4)}$. If we combine it with the standard quaternionic left-multiplication by unit vectors on $\mathbb{H}$, we have defined an $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ action on $P_{\mathrm{Spin}(4)} \times \mathbb{H}$. As it commutes with μ_-, it passes to the quotient $\slashed{S}_-(S^4)$.

By Lie group theory [51, Appendix B], we know that the 3-dimensional connected closed subgroups of $\mathrm{Sp}(2)$ are the lift of one of the following subgroups of $\mathrm{SO}(5)$:

$$\mathrm{SO}(3) \times \mathrm{Id}_2, \qquad \mathrm{Sp}(1) \times \mathrm{Id}_1,$$
$$\mathrm{SO}(3) \text{ acting irreducibly on } \mathbb{R}^5,$$

where $\mathrm{Sp}(1) \times \mathrm{Id}_1$ denotes both the subgroup acting on $\mathbb{H} \times \mathbb{R}$ by left multiplication and by right multiplication of the quaternionic conjugate. Observe that they are all diffeomorphic to $\mathrm{SU}(2)$. In particular, the family of 3-dimensional subgroups that do not sit diagonally in $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ consists of

$$G \times 1_{\mathrm{Sp}(1)} \subset \mathrm{Sp}(2) \times \mathrm{Sp}(1)$$

and

$$1_{\mathrm{Sp}(2)} \times \mathrm{Sp}(1) \subset \mathrm{Sp}(2) \times \mathrm{Sp}(1),$$

where G is one of the lifts above. These are going to be the subgroups of the automorphism group that we will take into consideration in Chapter 4.

2.4 Multi-moment maps

In [62] and [63], Madsen and Swann extended the classical notion of moment maps for symplectic manifolds to any closed geometry (M, α), i.e. a manifold X endowed with a closed form α. In order to recall the precise definition, we need to briefly discuss some properties of multi-vectors.

2.4.1 Cartan's extended formula

In this subsection we recall the basic definitions and properties of multi-vectors.

Definition 2.4.1. A multi-vector of degree k is a section of $\Gamma(\Lambda^k TM)$. A multi-vector of degree k, q, is simple if it has the form:

$$q = X_1 \wedge \ldots \wedge X_k,$$

for $X_i \in \Gamma(TM)$.

There is a natural extension of the interior product on differential forms which can be defined on simple multi-vectors and extended $\mathbb{R}$-linearly to the whole space:

$$\begin{aligned} i : \Gamma(\Lambda^k TM) \times \Gamma(\Lambda^r T^*M) &\to \Gamma(\Lambda^{(r-k)} T^*M) \\ (X_1 \wedge \ldots \wedge X_k, \alpha) &\mapsto i_{X_k} \circ \ldots \circ i_{X_1} \alpha \end{aligned}$$

Note that there is another important space describing k-tuples of tangent vectors: $\Lambda^k_{\mathbb{R}}(\Gamma(TM))$. It is important to observe that $\Lambda^k_{\mathbb{R}}(\Gamma(TM)) \neq \Gamma(\Lambda^k TM)$. Indeed, the former is much bigger than the latter, which is equal to $\Lambda^k_{C^\infty(M)}(\Gamma(TM))$. However, there is a natural $\mathbb{R}$-linear projection $\Lambda^k_{\mathbb{R}}(\Gamma(TM)) \to \Gamma(\Lambda^k TM)$ which can be defined on decomposable elements by:

$$X_1 \curlywedge \ldots \curlywedge X_k \mapsto X_1 \wedge \ldots \wedge X_k,$$

where $\curlywedge$ denotes the $\mathbb{R}$-wedge product. If $Q = X_1 \curlywedge \ldots \curlywedge X_k \in \Lambda^k_{\mathbb{R}}(\Gamma(TM))$ is decomposable, then, we can define:

$$\begin{aligned} Q_i &= (-1)^i X_1 \curlywedge \ldots \curlywedge X_{i-1} \curlywedge X_{i+1} \cdots \curlywedge X_k, \\ Q_{ij} &= (Q_i)_j. \end{aligned}$$

We also define the following useful operators which extend to the whole $\Lambda^k_{\mathbb{R}}(\Gamma(TM))$ by $\mathbb{R}$-linearity:

$$\begin{aligned} \mathcal{L}_Q \alpha &= \sum_{l=1}^{s} i_{Q_i} \mathcal{L}_{X_i} \alpha, \\ L(Q) &= \sum_{i,j} [X_i, X_j] \curlywedge Q_{ij}, \end{aligned}$$

where α is a given differential form.

Lemma 2.4.2 (Extended Cartan's formula; Madsen–Swann [63])**.** *Let $\alpha \in \Gamma(\Lambda^r T^*M)$ and let $p \in \Gamma(\Lambda^k TM)$. Then, for every $P \in \Lambda^k_{\mathbb{R}}(\Gamma(TM))$ projecting to p we have:*

$$i_p d\alpha - (-1)^k d(i_p \alpha) = \mathcal{L}_P \alpha - i_{L(P)} \alpha.$$

2.4.2 Multi-moment maps

Let $G \subset \mathrm{Aut}(M, \alpha)$ and let $\mathfrak{g}$ be the Lie algebra of G, which we identify in the usual way with the vector fields generated by G, i.e., $\mathfrak{g} \subset \Gamma(TM)$. Under this identification, we have $\Lambda^k \mathfrak{g} \subset \Lambda^k_{\mathbb{R}}(\Gamma(T^*M))$. Moreover, since the map sending an element of $\mathfrak{g}$ to a vector field is $\mathbb{R}$-linear, for every element of $\Lambda^k\mathfrak{g}$, we can associate a unique multi-vector. In other words, there is a natural inclusion of $\Lambda^k\mathfrak{g}$ in $\Lambda^k_{\mathbb{R}}(\Gamma(T^*M))$, and the projection to $\Gamma(\Lambda^k TM)$, when restricted to $\Lambda^k\mathfrak{g}$, is injective.

Observe that in this setting, the Extended Cartan's formula becomes:

$$(-1)^k d(i_p\alpha) = -i_{L(p)}\alpha,$$

for every $p \in \Lambda^k\mathfrak{g}$. This equation motivates the following definition.

Definition 2.4.3 (Madsen–Swann [63]). Let $\mathfrak{g}$ be a Lie algebra. The k^{th} Lie kernel of $\mathfrak{g}$ is:

$$\mathcal{P}_{\mathfrak{g},k} = \ker(L : \Lambda^k\mathfrak{g} \to \Lambda^{k-1}\mathfrak{g}).$$

Example 2.4.4. Here are some examples of k^{th} Lie kernels.

1. If G is abelian, $\mathcal{P}_{\mathfrak{g},k} = \Lambda^k\mathfrak{g}$ for every k,
2. For any Lie group $\mathcal{P}_{\mathfrak{g},1} = \mathfrak{g}$,
3. $\mathcal{P}_{\mathfrak{su}(2),3} = \Lambda^3\mathfrak{su}(2)$,
4. $\mathcal{P}_{\mathfrak{t}^2\times\mathfrak{su}(2),3} = \Lambda^3\mathfrak{su}(2) \oplus (\Lambda^2\mathfrak{t}^2 \otimes \Lambda^1\mathfrak{su}(2))$,
5. $\mathcal{P}_{\mathfrak{t}^2\times\mathfrak{su}(2),2} = \Lambda^2\mathfrak{t}^2 \oplus (\Lambda^1\mathfrak{t}^2 \otimes \Lambda^1\mathfrak{su}(2))$.

Remark 2.4.5. If (M, α) is a closed geometry of degree r, then $i_p\alpha$ is a closed form for every $p \in \mathcal{P}_{\mathfrak{g},r-1}$. Moreover, if $H^1(M) = \{0\}$, then there exists a function ν_p such that $d\nu_p = i_p\alpha$.

Definition 2.4.6 (Madsen–Swann [63]). Let (M, α) be a closed geometry of degree k, and let $G \subset \mathrm{Aut}(M, \alpha)$. A multi-moment map with respect to this action is an equivariant map $\nu : M \to \mathcal{P}^*_{\mathfrak{g},k-1}$ such that:

$$d\langle \nu, p\rangle = i_p\alpha,$$

for every $p \in \mathcal{P}_{\mathfrak{g},k-1}$.

Remark 2.4.7. For instance, the existence of multi-moment maps is granted if $H^1(M) = \{0\}$ and G is compact. This is always going to be our case for the rest of this thesis.

Multi-moment maps were particularly successful in the toric setting, where the k^{th} Lie kernel is trivial. Indeed, multi-moment maps were used to study toric G_2 manifolds [64], toric Spin(7) manifolds [65] and nearly–Kähler toric manifolds [26].

Chapter 3

Calibrated fibrations and linear calibrated vector bundles

In this chapter, we consider manifolds with exceptional holonomy that are fibred, in a suitable sense, by coassociative or Cayley submanifolds.

As a first step, we will give a general definition of locally trivial calibrated fibrations in a Riemannian manifold (M, g) endowed with a calibration α. These objects are fibre bundles whose fibres are calibrated submanifolds. The Riemannian structure, g, induces an Ehresmann connection on the bundle. This splitting extends to the algebra of differential forms and, hence, to the condition $d\alpha = 0$. The process was reversed in the cases of our interest by Donaldson in [27] (cfr. Proposition 3.1.2 and Proposition 3.1.3).

The definition of calibrated fibration that we will use in Chapter 4 and Chapter 5 is adapted, to a general setting, from the definition of coassociative fibrations introduced by Karigiannis and Lotay in [49]. Such a definition coincides with the locally trivial one in a open dense set, and allows the fibres to be singular and/or intersect in the complement of this open dense set.

Afterwards, we turn our attention to a special class of coassociative and Cayley fibrations, which we call linear coassociative fibrations and linear Cayley fibrations. These objects are locally trivial calibrated fibrations on a Euclidean vector bundle, endowed with a compatible linear connection and G_2 or Spin(7) structure. Under an isotropic condition, we are able to explicitely solve the systems of PDEs corresponding to the torsion-free condition. All the solutions turned out to be deformations of the Bryant–Salamon manifolds described in Section 2.2.4 and Section 2.3.3.

3.1 Calibrated fibrations in manifolds of exceptional holonomy

3.1.1 Definitions of calibrated fibrations

The obvious way to define a calibrated fibration is as follows.

Definition 3.1.1. Let (M, g) be an n-dimensional manifold with a k-dimensional calibration α, $k \leq n$. M admits a locally trivial α-calibrated fibration if there exists a $(n-k)$-dimensional manifold B and a smooth fibre bundle structure $\pi : M \to B$ such that $\pi^{-1}(b)$ is an α-calibrated submanifold of M for every $b \in B$.

Given a locally trivial calibrated fibration $\pi : M \to B$, we can use the Riemannian metric g to define an Ehresmann connection H on M. This means that we have a splitting of $TM = H \oplus V$, where V is the tangent bundle along the fibres. Moreover, the splitting propagates to the space of k-forms:

$$\Gamma(\Lambda^k T^* M) = \bigoplus_{p,q \geq 0, p+q=k} \Gamma^{p,q}.$$

Under this decomposition, the exterior differential splits into $d = d_F + d_H + F_H$, where:

$$\begin{aligned} d_F &: \Gamma^{p,q} \to \Gamma^{p,q+1}, \\ d_H &: \Gamma^{p,q} \to \Gamma^{p+1,q}, \\ F_H &: \Gamma^{p,q} \to \Gamma^{p+2,q-1}. \end{aligned}$$

The condition that the fibres of the bundle are α-calibrated and $d\alpha = 0$, gives a system of PDEs. In the case of our interest we can reverse this procedure as follows.

Proposition 3.1.2 (Donaldson [27]). *Let $\pi : M \to B$ be a fibre bundle with 7-dimensional total space M and 3-dimensional base space B. A G_2-structure on M with coassociative fibres diffeomorphic to F, is equivalent to the following data:*

- *a connection H, identified with its horizontal distribution,*
- *an $\omega \in \Gamma^{(1,2)}$ such that, at each point, ω can be viewed as an injection of H to a maximal negative subspace for the wedge product,*
- *a tensor $\lambda \in \Gamma^{(3,0)}$ such that the value of λ is positive at each point, regarded as an element of $\Gamma(\Lambda^3 T^* B)$.*

Explicitly, the induced G_2*-structure is* $\varphi = \lambda + \omega$*, and the torsion-free condition becomes the following system of PDEs:*

$$\begin{cases} d_F\omega = 0 \\ d_H\omega = 0, \\ d_F\lambda = -F_H\omega, \\ d_H\mu = 0, \\ d_F\Theta = -F_H\mu, \\ d_H\Theta = 0, \end{cases} \tag{3.1.1}$$

where μ *and* Θ *are determined algebraically from* ω *and* λ *(cfr. [27, Lemma 3]).*

Proposition 3.1.3 (Donaldson [27])**.** *Let* $\pi : M \to B$ *be a fibre bundle, with* M *8-dimensional and* B *4-dimensional. A* $\mathrm{Spin}(7)$*-structure on* M *with Cayley fibres diffeomorphic to* F*, is equivalent to the following data:*

- *a connection* H*, identified with its horizontal distribution,*
- *a tensor* $\lambda \in \Gamma^{(4,0)}$ *such that the value of* λ *is positive at each point, regarded as an element of* $\Gamma(\Lambda^4 T^*B)$*,*
- *a tensor* $\mu \in \Gamma^{(0,4)}$ *such that the value of* μ *is positive at each point, regarded as an element of* $\Gamma(\Lambda^4 T^*F)$*,*
- *a tensor* $\nu \in \Gamma^{(2,2)}$ *such that at each point* ν *can be viewed as an isomorphism between anti-self-dual two forms of* H *and* V*.*

Explicitly, the induced $\mathrm{Spin}(7)$*-structure is* $\Phi = \lambda + \mu + \nu$*, and the torsion-free condition becomes the following system of PDEs:*

$$\begin{cases} d_F\lambda + F_H\nu = 0, \\ d_H\mu = 0, \\ d_H\nu = 0, \\ d_F\nu + F_H\mu = 0. \end{cases} \tag{3.1.2}$$

These propositions are a consequence of the local model for G_2 and $\mathrm{Spin}(7)$ manifolds and from splitting the torsion-free condition ($d\varphi = d * \varphi = 0$ and $d\Phi = 0$, respectively) with respect to the connection H.

In Chapters 4 and 5, we will need a different definition for calibrated fibration, which allows singular and intersecting fibres. The reason behind this looser definition comes from physics, where it is important to let the fibres to be singular, and from the following proposition:

Proposition 3.1.4 (Baraglia [8]). *There are no locally trivial coassociative fibrations on compact manifolds with full holonomy* G_2.

The definition of calibrated fibration that we will make use of is:

Definition 3.1.5. Let (M, g) be an n-dimensional manifold with a k-dimensional calibration α, $k \leq n$. M admits a α-calibrated fibration if there exists a family of α-calibrated submanifolds N_b (possibly singular) parametrized by a $(n-k)$-dimensional space $\mathcal{B}$ satisfying the following properties:

- M is covered by the family $\{N_b\}_{b \in \mathcal{B}}$;
- there exists an open dense set $\mathcal{B}^\circ \subset \mathcal{B}$ such that N_b is smooth for all $b \in \mathcal{B}^\circ$;
- there exists an open dense set $M' \subset M$, a submanifold $\mathcal{B}' \subset \mathcal{B}$ and a smooth fibre bundle $\pi : M' \to \mathcal{B}'$ with fibre N_b for all $b \in \mathcal{B}'$.

Remark 3.1.6. The last point allows the α-calibrated submanifolds in the family $\mathcal{B}$ to intersect. Indeed, this may happen in $M \setminus M'$. Moreover, we may lose information (e.g. completeness and topology) when we restrict the fibres to M'.

Remark 3.1.7. It is clear that a locally trivial calibrated fibration is, in particular, a calibrated fibration with $\mathcal{B}^\circ = \mathcal{B}$ and $M' = M$.

3.2 Linear coassociative fibrations

In this section, we study a special case of coassociative fibrations. Namely, locally trivial coassociative fibrations with a compatible vector bundle and G_2-structure.

Definition 3.2.1. Let (M, φ) be a G_2 manifold and let B a 3-dimensional manifold. A locally trivial coassociative fibration $\pi : M \to B$ is a linear coassociative fibration if the following conditions are satisfied:

- $\pi : M \to B$ has the structure of a Euclidean vector bundle,
- the induced connection H is linear, i.e. it is induced by a covariant derivative ∇, and it is compatible with the Euclidean structure,

- at every point of B there exists a local coframe $\{b_1, b_2, b_3\}$ of B and a local orthonormal trivialization of the bundle $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ such that the G_2-structure on M and relative Hodge dual are respectively given by:

$$\begin{aligned}\varphi :=& f_{123} b_1 \wedge b_2 \wedge b_3 + \sum_{i=1}^{3} f_i b_i \wedge (g_{0i}\xi_0 \wedge \xi_i - g_{jk}\xi_j \wedge \xi_k),\\ *\varphi =& g_{0123}\xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 - \sum_{i=1}^{3} f_{jk} b_j \wedge b_k \wedge (g_{0i}\xi_0 \wedge \xi_i - g_{jk}\xi_j \wedge \xi_k),\end{aligned}$$

  where f_i, g_i are smooth positive functions on M, (i,j,k) is a positive permutation of $(1,2,3)$ and $\{\xi_0, \xi_1, \xi_2, \xi_3\}$ is the dual of the basis induced by the σ_is on the vertical space.

Example 3.2.2. The local model $\mathbb{R}^7 \cong \mathbb{R}^3 \oplus \mathbb{R}^4$, as described in Section 2.2.1, is a linear coassociative fibration. In this case, the connection is trivial. A nontrivial example consists of the Bryant–Salamon manifold of topology $\not{S}(S^3)$. The linear connection is simply the spin connection in this setting.

3.2.1 The system of PDEs for linear coassociative fibrations

We now want to write explicitly the local system of PDEs for the torsion-free condition in the linear coassociative fibrations case. To this end, let $\{e_1, e_2, e_3\}$ be a the dual frame of the coframe $\{b_1, b_2, b_3\}$ given in Definition 3.2.1. The coframe induces a metric on B, $g_B := \sum_{i=1}^{3} b_i^2$, and the relative Levi-Civita connection induces the connection matrix $\{\omega_i^j\} \in \mathfrak{so}(n)$, which satisfies the structure equations:

$$\begin{aligned}d\underline{b} &= -\omega \wedge \underline{b},\\ R &= d\omega + \omega \wedge \omega,\end{aligned}$$

where $R \in \mathfrak{so}(n)$ denotes the curvature 2-form.

Let $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ be a local orthonormal trivialization of the Euclidean bundle as in Definition 3.2.1, inducing the parametrization of the fibres: $a_0\sigma_0 + ... + a_3\sigma_3$. Since the connection, H, is compatible with the Euclidean structure, we get the associated connection matrix of 1-forms $\{A_i^j\} \subset \mathfrak{so}(n)$ and the curvature matrix of 2-forms $\{F_i^j\} \subset \mathfrak{so}(n)$. These matrices of differential forms are related by the structural equation:

$$F = dA + A \wedge A.$$

It is well-known that the dual of the horizontal and the vertical spaces are spanned, respectively, by:

$$b_i := \pi^*(b_i), \qquad \qquad \xi_l := da_l + \sum_{m=0}^{3} A^l_m a_m,$$

for $i = 1, ..., 3$ and $l = 0, ..., 3$.

Remark 3.2.3. From now on, as an abuse of notation, we will omit the pullback symbol from our discussion. Moreover, products of functions denoted by the same letter, but with different subscripts, will just be written with repeated subscripts, e.g., if f_1, f_2, f_3 are smooth functions, then, $f_1 \cdot f_2 = f_{12}$ and $f_1 \cdot f_2 \cdot f_3 = f_{123}$.

From a straightforward computation, we can write the system of PDEs explicitly.

Proposition 3.2.4. *Let $\pi : M \to B$ be a linear coassociative fibration. The torsion-free condition becomes, in a local trivialization as in Definition 3.2.1, the following system of PDEs:*

$$\begin{cases} 0 = \sum_{i=1}^3 \left(\sum_{l=0}^3 \partial_{a_l}(f_i g_{0i})\xi_l \wedge \xi_0 \wedge \xi_i - \partial_{a_l}(f_i g_{jk})\xi_l \wedge \xi_j \wedge \xi_k\right) \\ 0 = \sum_{i=1}^3 \big(\sum_{l=1}^3 e_l(f_i g_{01}) b_l \wedge b_i \wedge \xi_0 \wedge \xi_i - e_l(f_i g_{jk}) b_l \wedge b_i \wedge \xi_j \wedge \xi_k \\ \qquad\qquad - \sum_{l,m=0}^3 \partial_{a_l}(f_i g_{0i}) a_m A^l_m \wedge b_i \wedge \xi_0 \wedge \xi_i - \partial_{a_l}(f_i g_{jk}) a_m A^l_m \wedge b_i \wedge \xi_j \wedge \xi_k \\ \qquad\qquad + \sum_{l=1}^3 f_i g_{0i}(-\omega^i_l \wedge b_l \wedge \xi_0 \wedge \xi_i) - f_i g_{jk}(-\omega^i_l \wedge b_l \wedge \xi_j \wedge \xi_k) \\ \qquad\qquad - \sum_{l=0}^3 f_i g_{0i} b_i \wedge (\xi_l \wedge A^0_l \wedge \xi_i - \xi_0 \wedge \xi_l \wedge A^i_l) \\ \qquad\qquad - \sum_{l=0}^3 -f_i g_{jk} b_i \wedge (\xi_l \wedge A^j_l \wedge \xi_k - \xi_j \wedge \xi_l \wedge A^k_l)\big) \\ 0 = \sum_{l=0}^3 \partial_{a_l}(f_{123})\xi_l \wedge b_1 \wedge b_2 \wedge b_3 \\ \qquad - \sum_{i=1}^3 f_i g_{01} b_i \wedge (F^0_{\underline{a}} \wedge \xi_i - F^i_{\underline{a}} \wedge \xi_0) - f_i g_{jk} b_i \wedge (F^j_{\underline{a}} \wedge \xi_k - F^k_{\underline{a}} \wedge \xi_j) \\ 0 = \left(\sum_{l=1}^3 e_l(g_{0123}) b_l - \sum_{l,m=0}^3 \partial_{a_l}(g_{0123}) a_m A^l_m \right) \wedge \xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 \\ 0 = \sum_{i=1}^3 \sum_{l=0}^3 \partial_{a_l}(f_{jk} g_{0i})\xi_l \wedge b_j \wedge b_k \wedge \xi_0 \wedge \xi_i - \partial_{a_l}(f_{jk} g_{jk})\xi_l \wedge b_j \wedge b_k \wedge \xi_j \wedge \xi_k \\ \qquad - g_{0123} \sum_{l=0}^3 (-1)^l F^l_{\underline{a}} \wedge \xi_0 \wedge ... \wedge \hat{\xi_l} \wedge ... \wedge \xi_4 \\ 0 = \sum_{i=1}^3 \sum_{l=1}^3 e_l(f_{jk} g_{0i}) b_l \wedge b_j \wedge b_k \wedge \xi_0 \wedge \xi_i - e_l(f_{jk} g_{jk}) b_l \wedge b_j \wedge b_k \wedge \xi_j \wedge \xi_k \\ \qquad\qquad + \sum_{l,m=0}^3 \partial_{a_l}(f_{jk} g_{0i}) a_m A^l_m \wedge b_j \wedge b_k \wedge \xi_0 \wedge \xi_i \\ \qquad\qquad - \sum_{l,m=0}^3 \partial_{a_l}(f_{jk} g_{jk}) a_m A^l_m \wedge b_j \wedge b_k \wedge \xi_j \wedge \xi_k \\ \qquad\qquad - \sum_{l=1}^3 f_{jk}(\omega^j_l \wedge b_l \wedge b_k - b_j \wedge \omega^k_l \wedge b_l) \wedge (g_{01}\xi_0 \wedge \xi_1 - g_{jk}\xi_j \wedge \xi_k) \\ \qquad\qquad + \sum_{l=0}^3 f_{jk} g_{0i} b_j \wedge b_k \wedge (\xi_l \wedge A^0_l \wedge \xi_i - \xi_0 \wedge \xi_l \wedge A^i_l) \\ \qquad\qquad - \sum_{l=0}^3 f_{jk} g_{jk} b_j \wedge b_k \wedge (\xi_l \wedge A^j_l \wedge \xi_k - \xi_j \wedge \xi_l \wedge A^k_l) \end{cases}, \tag{3.2.1}$$

where $F^l_{\underline{a}} := \sum_{m=0}^3 F^l_m a_m$.

Example 3.2.5. We now give a local example of a linear coassociative fibration. Let B be a 3-manifold of constant sectional curvature k with local orthonormal coframe $\{b_1, b_2, b_3\}$. Let E be a Euclidean vector 4-bundle with local orthonormal sections, $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$, and let A be the relative metric compatible connection:

$$A = \begin{pmatrix} 0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_1 & 0 & -\alpha_3 - 2\rho_3 & \alpha_2 + 2\rho_2 \\ \alpha_2 & \alpha_3 + 2\rho_3 & 0 & -\alpha_1 - 2\rho_1 \\ \alpha_3 & -\alpha_2 - 2\rho_2 & \alpha_1 + 2\rho_1 & 0, \end{pmatrix},$$

where $\rho_i = \frac{1}{2}\omega_j^k$ and the α_i are such that A is compatible with the curvature matrix:

$$F = -\frac{k}{2} \begin{pmatrix} 0 & b_2 \wedge b_3 & b_3 \wedge b_1 & b_1 \wedge b_2 \\ -b_2 \wedge b_3 & 0 & -b_1 \wedge b_2 & b_3 \wedge b_1 \\ -b_3 \wedge b_1 & b_1 \wedge b_2 & 0 & -b_2 \wedge b_3 \\ -b_1 \wedge b_2 & -b_3 \wedge b_1 & b_2 \wedge b_3 & 0 \end{pmatrix},$$

i.e. $F = dA + A \wedge A$. If $\{\xi_0, \xi_1, \xi_2, \xi_3\}$ are the vertical 1-forms dual to the basis induced by the σ_is on the vertical space, then, it is straightforward to check that the G_2-structure:

$$\varphi = f^3 b_1 \wedge b_2 \wedge b_3 + fg^2 \sum_{i=1}^{3} b_i \wedge (\xi_0 \wedge \xi_i - \xi_j \wedge \xi_k)$$

is torsion-free when f, g are functions only depending on the square of the distance from the zero section, r^2, and satisfying the ODEs in r^2:

$$(\dot{fg^2}) = 0, \quad (\dot{f^3}) = \frac{3k}{4} fg^2, \quad g^4 \frac{k}{4} = (\dot{f^2 g^2}).$$

We call such examples deformed G_2 Bryant–Salamon manifolds. Indeed, if we pick $\alpha_i = -\rho_i$, then, E becomes the spinor bundle over B and we recover the Bryant–Salamon manifolds of topology $\not{S}(S^3)$ (cfr. Remark 2.2.19).

A similar idea was employed by Herfray–Krasnov–Scarinaci–Shtanov in [38] for the Bryant–Salamon manifolds of topology $\Lambda^2_-(X)$.

3.2.2 Isotropic linear coassociative fibrations

Even though Eq. (3.2.1) is a complicated system of PDEs, under some isotropic condition we are able to find all the solutions of the system.

Theorem 3.2.6. *Let (M, φ) be a linear coassociative fibration such that the horizontal and vertical spaces are isotropic, i.e., for every $p \in M$ and unit $v, w \in H_p \subset T_pM$ (or $V_p \subset T_pM$) there exists a local isomorphism F of (M, φ) such that $dF_p(v) = w$. Then, (M, φ) is locally isomorphic to a deformed G_2 Bryant–Salamon manifold.*

Proof. Under the isotropic condition, we have $f := f_i$ for all $i = 1, 2, 3$ and $g := g_l$ for all $l = 0, 1, 2, 3$. Hence, the system of Eq. (3.2.1) becomes:

$$
\begin{cases}
0=\partial_{a_l}(fg^2)\sum_{i=1}^3\left(\sum_{l=0}^3\xi_l\wedge\xi_0\wedge\xi_i-\xi_l\wedge\xi_j\wedge\xi_k\right)\\
0=\sum_{i=1}^3\Big(\sum_{l=1}^3 e_l(fg^2)b_l\wedge b_i\wedge(\xi_0\wedge\xi_i-\xi_j\wedge\xi_k)\\
\qquad\qquad-\sum_{l,m=0}^3\partial_{a_l}(fg^2)a_mA_m^l\wedge b_i\wedge(\xi_0\wedge\xi_i-\xi_j\wedge\xi_k)\\
\qquad\qquad-\sum_{l=1}^3 fg^2\omega_l^i\wedge b_l\wedge(\xi_0\wedge\xi_i-\xi_j\wedge\xi_k)\\
\qquad\qquad-\sum_{l=0}^3 fg^2b_i\wedge(\xi_l\wedge A_l^0\wedge\xi_i-\xi_0\wedge\xi_l\wedge A_l^i-\xi_l\wedge A_l^j\wedge\xi_k+\xi_j\wedge\xi_l\wedge A_l^k)\Big)\\
0=\sum_{l=0}^3\partial_{a_l}(f^3)\xi_l\wedge b_1\wedge b_2\wedge b_3\\
\qquad-\sum_{i=1}^3 fg^2b_i\wedge(F_{\underline{a}}^0\wedge\xi_i-F_{\underline{a}}^i\wedge\xi_0-F_{\underline{a}}^j\wedge\xi_k+F_{\underline{a}}^k\wedge\xi_j)\\
0=\left(\sum_{l=1}^3 e_l(g^4)b_l-\sum_{l,m=0}^3\partial_{a_l}(g^4)a_mA_m^l\right)\wedge\xi_0\wedge\xi_1\wedge\xi_2\wedge\xi_3\\
0=\sum_{i=1}^3\sum_{l=0}^3\partial_{a_l}(f^2g^2)\xi_l\wedge b_j\wedge b_k\wedge(\xi_0\wedge\xi_i-\xi_j\wedge\xi_k)\\
\qquad-g^4\sum_{l=0}^3(-1)^lF_{\underline{a}}^l\wedge\xi_0\wedge\ldots\wedge\hat{\xi_l}\wedge\ldots\wedge\xi_4\\
0=\sum_{i=1}^3(\sum_{l=1}^3 e_l(f^2g^2)b_l+\sum_{l,m=0}^3\partial_{a_l}(f^2g^2)a_mA_m^l)\wedge b_j\wedge b_k\wedge(\xi_0\wedge\xi_i-\xi_j\wedge\xi_k)\\
\qquad\qquad-\sum_{l=0}^3 f^2g^2(\omega_l^j\wedge b_l\wedge b_k-b_j\wedge\omega_l^k\wedge b_l)\wedge(\xi_0\wedge\xi_i-\xi_j\wedge\xi_k)\\
\qquad\qquad+\sum_{l=0}^3 f^2g^2b_j\wedge b_k\wedge(\xi_l\wedge A_l^0\wedge\xi_i-\xi_0\wedge\xi_l\wedge A_l^i)\\
\qquad\qquad-\sum_{l=0}^3 f^2g^2b_j\wedge b_k\wedge(\xi_l\wedge A_l^j\wedge\xi_k-\xi_j\wedge\xi_l\wedge A_l^k)
\end{cases}
.
\tag{3.2.2}
$$

We begin our analysis from the fifth equation, which becomes, after regrouping:

$$-g^4F_{\underline{a}}^0=\sum_{i=1}^3\partial_{a_i}(f^2g^2)b_j\wedge b_k, \tag{3.2.3}$$

$$-g^4F_{\underline{a}}^i=\partial_{a_k}(f^2g^2)b_k\wedge b_i-\partial_{a_j}(f^2g^2)b_i\wedge b_j-\partial_{a_0}(f^2g^2)b_j\wedge b_k, \tag{3.2.4}$$

where, as usual, (i,j,k) is any cyclic permutation of (1,2,3). As $\{F_j^i\}\in\mathfrak{so}(4)$, we can see that:

$$0=g^4F_{\underline{a}}^{\underline{a}}=-\sum_{i=1}^3 E_i(f^2g^2)b_j\wedge b_k,$$

where the E_is are the generators of $\mathfrak{sp}(1)$:

$$
\begin{aligned}
E_1&=a_1\partial_{a_0}-a_0\partial_{a_1}-a_3\partial_{a_2}+a_2\partial_{a_3},\\
E_2&=a_2\partial_{a_0}+a_3\partial_{a_1}-a_0\partial_{a_2}-a_1\partial_{a_3},\\
E_3&=a_3\partial_{a_0}-a_2\partial_{a_1}+a_1\partial_{a_2}-a_0\partial_{a_3}.
\end{aligned}
$$

As a consequence of this, we deduce that f^2g^2 only depends on $r^2=a_0^2+a_1^2+a_2^2+a_3^3$ when restricted to a fibre, and hence, $\partial_{a_i}(f^2g^2)=2a_i(\dot{f^2g^2})$, where the dot denotes the derivative with respect to r^2. Another way to rewrite Eq. (3.2.3) and Eq. (3.2.4) in terms of the E_is is:

$$-g^4F_{\underline{a}}^{E_i}=2r^2(\dot{f^2g^2})b_j\wedge b_k, \tag{3.2.5}$$

which holds for every $i = 1, 2, 3$. Observe that Eq. (3.2.5) implies that $F^{E_1}_{\underline{a}23} = F^{E_2}_{\underline{a}31} = F^{E_3}_{\underline{a}12}$. Now, since one can easily verify from the antisymmetry of F that $E_i(F^{E_i}_{\underline{a}jk}) = 0$, we deduce that $F^{E_i}_{\underline{a}jk}$ depends only on r^2, when restricted to any fibre. Therefore, Eq. (3.2.5) implies that g, and consequently f, has the same property. Finally, we deduce that the curvature form is:

$$F = -\frac{k}{2}\begin{pmatrix} 0 & b_2 \wedge b_3 & b_3 \wedge b_1 & b_1 \wedge b_2 \\ -b_2 \wedge b_3 & 0 & -b_1 \wedge b_2 & b_3 \wedge b_1 \\ -b_3 \wedge b_1 & b_1 \wedge b_2 & 0 & -b_2 \wedge b_3 \\ -b_1 \wedge b_2 & -b_3 \wedge b_1 & b_2 \wedge b_3 & 0 \end{pmatrix},$$

where $k := -2F^0_{123} = -2F^0_{231} = -2F^0_{312}$. The functions, f, g and k satisfy:

$$g^4 \frac{k}{4} = (\dot{f^2 g^2}). \tag{3.2.6}$$

We now turn our attention to the other equations. The first and the third ones read, respectively:

$$(\dot{f g^2}) = 0, \tag{3.2.7}$$

$$(\dot{f^3}) = \frac{3k}{4} f g^2. \tag{3.2.8}$$

The fourth one implies that g is independent from the basis B. Indeed, A being a metric connection implies $\sum_{l,m} (\dot{g^4}) a_l a_m A^l_m = 0$.

Before studying the remaining two equations of Eq. (3.2.2), we make the following assumption, which we will prove later.

Claim: Up to changing the metric g_B, f can be assumed to be independent from the basis B.

The claim, together with Eq. (3.2.6), implies that k is a constant. Moreover, since $e_i(f) = 0$ for every $i = 1, 2, 3$, we deduce that the two remaining equations become:

$$\begin{aligned} 0 =& -(\omega^j_i + A^k_0 - A^j_i) \wedge b_k \wedge b_i + (\omega^i_k + A^j_0 - A^i_k) \wedge b_i \wedge b_j, \\ 0 =& -2(\omega^j_i + A^k_0 - A^j_i) \wedge b_j + (\omega^i_k + A^j_0 - A^i_k) \wedge b_k, \end{aligned}$$

for every (i, j, k) cyclic permutation of $(1, 2, 3)$. Therefore, one can verify that:

$$\omega^j_i + A^k_0 - A^j_i = 0. \tag{3.2.9}$$

Combing the structure equations for the curvature forms, R, F, with Eq. (3.2.9), we obtain:

$$R^i_j = F^i_0 - F^k_j = k b_i \wedge b_j.$$

Hence, (B, g_B) has constant sectional curvature k and we conclude.

Proof of the claim: The second (or, analogously, the sixth) equation of Eq. (3.2.2), implies that $e_l(f^2)/f^2 = e_l(\log(f^2))$ is independent from the fibres, hence, f^2 can be rewritten as the product of f_B^2, depending only on B, and f_F^2 depending only on the fibres. Changing the metric on B such that $\{f_B b_i\}_i$ form an orthonormal frame, we can reabsorb $e_l(f^2)$ into the connection term and assume f independent from the base. □

□

Remark 3.2.7. Observe that the isotropy condition rules out well-known examples coming from lower dimensional geometries. For instance, one can notice the product of a Calabi–Yau manifold with a flat $\mathbb{R}$ is a G_2-manifold of the form described in Proposition 3.2.4. However, the f_i or g_i corresponding to the flat direction would need to be constant. This clearly gives a contradiction unless we are in the flat local model, where they all coincide.

3.3 Linear Cayley fibrations

Similarly to linear coassociative fibrations, we consider locally trivial Cayley fibrations with a compatible vector bundle and Spin(7)-structure.

Definition 3.3.1. Let (M, Φ) be a Spin(7) manifold and let B a 4-dimensional manifold. A locally trivial Cayley fibration $\pi : M \to B$ is a linear Cayley fibration if the following conditions are satisfied:

- $\pi : M \to B$ has the structure of a Euclidean vector bundle,
- the connection H is linear, i.e., it is induced from a covariant derivative ∇, and it is compatible with the Euclidean structure,
- at every point of B there exists a local coframe $\{b_0, b_1, b_2, b_3\}$ of B and a local orthonormal trivialization of the bundle $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ such that the Spin(7)-structure on M is given by:

$$\begin{aligned}\Phi :=& f_{0123} b_0 \wedge b_1 \wedge b_2 \wedge b_3 + g_{0123} \xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 + \\ &+ \sum_{i=1}^{3} (g_{0i} \xi_0 \wedge \xi_i - g_{jk} \xi_j \wedge \xi_k) \wedge (f_{0i} b_0 \wedge b_i - f_{jk} b_j \wedge b_k),\end{aligned}$$

  where f_i, g_i are smooth positive functions on M and (i, j, k) is a positive permutation of $(1, 2, 3)$ and $\{\xi_0, \xi_1, \xi_2, \xi_3\}$ is the dual of the basis induced by the σ_i on the vertical space.

Example 3.3.2. The local model $\mathbb{R}^8 \cong \mathbb{R}^4 \oplus \mathbb{R}^4$, as described in Section 2.3.1, is a linear Cayley fibration. The product of a flat $\mathbb{R}^4$ with an hyperkähler manifold is a linear Cayley fibration. In these cases, the connection is trivial.

A nontrivial example consists of the Bryant–Salamon manifolds $\cancel{S}\_(S^4)$. The linear connection is simply the spin connection in this setting.

3.3.1 The system of PDEs for linear Cayley fibrations

As in Section 3.2.1, we rewrite the local system of PDEs for the torsion-free condition in the linear Cayley fibrations case. Let $\{b_0, b_1, b_2, b_3\}$ be as in Definition 3.3.1 with dual frame $\{e_0, e_1, e_2, e_3\}$. The frame induces a metric on B, $g_B := \sum_{i=0}^{3} b_i^2$, and hence a relative Levi-Civita connection form $\{\omega_i^j\}$. This connection form and the relative curvature 2-form, $R \in \mathfrak{so}(n)$, satisfy the structure equations:

$$
\begin{aligned}
d\underline{b} &= -\omega \wedge \underline{b}, \\
R &= d\omega + \omega \wedge \omega.
\end{aligned}
$$

We parametrize the fibres by $a_0\sigma_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$, where the σ_is are as in Definition 3.3.1. If $\{A_i^j\} \in \mathfrak{so}(n)$ is the connection matrix with respect to this trivialization, then the curvature matrix is $\{F_j^i = dA_j^i + \sum_l A_l^i \wedge A_m^l\}$.

The dual of the horizontal and the vertical spaces are spanned, respectively, by:

$$
b_i := \pi^*(b_i), \qquad \xi_i := da_i + \sum_{l=0}^{n} A_l^i a_l,
$$

for $i = 0, ..., 3$.

An explicit computation, similar to the one for Proposition 3.2.4, gives the system of PDEs for the torsion-free condition.

Proposition 3.3.3. *Let $\pi : M \to B$ be a linear Cayley fibration. The torsion-free condition becomes, in a local trivialization as in Definition 3.3.1, the following system of*

PDEs:

$$
\begin{cases}
0=\sum_{l=0}^{3}\partial_{a_l}(f_{0123})\xi_l\wedge \mathrm{vol}_H \\
\quad +\sum_{i=1}^{3}\tilde{\Omega}_i\wedge\big(g_{0i}(F^0_{\underline{a}}\wedge\xi_i-F^i_{\underline{a}}\wedge\xi_0)-g_{jk}(F^j_{\underline{a}}\wedge\xi_k-F^k_{\underline{a}}\wedge\xi_j)\big) \\
0=\sum_{l=0}^{3}\big(e_l(g_{0123})b_l-\partial_{a_l}(g_{0123})\sum_{m=0}^{3}A^l_m a_m\big)\wedge\mathrm{vol}_V \\
0=\sum_{i=1}^{3}\big(\sum_{l,m=0}^{3}(\partial_{a_l}f_{jk})a_mA^l_m\wedge b_j\wedge b_k-\sum_{l,m=0}^{3}(\partial_{a_l}f_{0i})a_mA^l_m\wedge b_0\wedge b_i\big)\wedge\Upsilon_i+ \\
\quad\quad +\big(\sum_{l,m=0}^{3}(\partial_{a_l}g_{jk})a_mA^l_m\wedge\xi_j\wedge\xi_k-\sum_{l,m=0}^{3}(\partial_{a_l}g_{0i})a_mA^l_m\wedge\xi_0\wedge\xi_i\big)\wedge\tilde{\Omega}_i+ \\
\quad\quad +\big(\sum_{l=0}^{3}(e_lg_{0i})b_l\wedge\xi_0\wedge\xi_i-\sum_{l=0}^{3}(e_lg_{jk})b_l\wedge\xi_j\wedge\xi_k\big)\wedge\tilde{\Omega}_i+ \\
\quad\quad +\big(\sum_{l=0}^{3}(e_lf_{0i})b_l\wedge b_0\wedge b_i-\sum_{l=0}^{3}(e_lf_{jk})b_l\wedge b_j\wedge b_k\big)\wedge\Upsilon_i+ \\
\quad\quad +\big((\omega^k_if_{0i}-\omega^0_jf_{jk})b_0\wedge b_k-(\omega^k_if_{jk}-\omega^0_jf_{0i})b_i\wedge b_j)\big)\wedge\Upsilon_i \\
\quad\quad +\big((\omega^0_kf_{jk}-\omega^i_jf_{0i})b_0\wedge b_j-(\omega^0_kf_{0i}-\omega^i_jf_{jk})b_k\wedge b_i\big)\wedge\Upsilon_i \\
\quad\quad +\big((A^k_ig_{0i}-A^0_jg_{jk})\xi_0\wedge\xi_k-(g_{jk}A^k_i-A^0_jg_{0i})\xi_i\wedge\xi_j)\big)\wedge\tilde{\Omega}_i \\
\quad\quad +\big((A^0_kg_{jk}-A^i_jg_{0i})\xi_0\wedge\xi_j-(A^0_kg_{0i}-A^i_jg_{jk})\xi_k\wedge\xi_i\big)\wedge\tilde{\Omega}_i \\
0=\sum_{i=1}^{3}\big(\sum_{l=0}^{3}\big((\partial_{a_l}f_{0i})b_0\wedge b_i-(\partial_{a_l}f_{jk})b_j\wedge b_k\big)\wedge\xi_l\big)\wedge\Upsilon_i \\
\quad\quad +\big(\sum_{l=0}^{3}\big((\partial_{a_l}g_{0i})\xi_0\wedge\xi_i-(\partial_{a_l}g_{jk})\xi_j\wedge\xi_k\big)\wedge\xi_l\big)\wedge\tilde{\Omega}_i \\
\quad +g_{0123}\sum_{l=0}^{3}(F^l_{\underline{a}}\wedge\xi_0\wedge\ldots\wedge\hat{\xi}_l\wedge\ldots\wedge\xi_4)
\end{cases}, \tag{3.3.1}
$$

where $F^l_{\underline{a}}:=\sum_{m=0}^{3}F^l_ma_m$, $\mathrm{vol}_H=b_0\wedge b_1\wedge b_2\wedge b_3$, $\tilde{\Omega}_i:=f_{0i}b_0\wedge b_i-f_{jk}b_j\wedge b_k$ *and* $\Upsilon_i:=g_{0i}\xi_0\wedge\xi_i-g_{jk}\xi_j\wedge\xi_k$.

Example 3.3.4. We now give a local example of linear Cayley fibration. Let B be a self-dual, Einstein 4-manifold of scalar curvature k with local orthonormal coframe $\{b_0,b_1,b_2,b_3\}$. Consider on B the 1-forms ρ_i, for $i=1,2,3$ characterized by the equation:

$$
d\begin{pmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\end{pmatrix}=-\begin{pmatrix}0&-2\rho_3&2\rho_2\\ 2\rho_3&0&-2\rho_1\\ -2\rho_2&2\rho_1&0\end{pmatrix}\wedge\begin{pmatrix}\Omega_1\\ \Omega_2\\ \Omega_3\end{pmatrix},
$$

where $\Omega_i=b_0\wedge b_i-b_j\wedge b_k$. Let E be a Euclidean vector 4-bundle with local orthonormal sections, $\{\sigma_0,\sigma_1,\sigma_2,\sigma_3\}$, and let A be the metric compatible connection:

$$
A=\begin{pmatrix}0&-\alpha_1&-\alpha_2&-\alpha_3\\ \alpha_1&0&-\alpha_3-2\rho_3&\alpha_2+2\rho_2\\ \alpha_2&\alpha_3+2\rho_3&0&-\alpha_1-2\rho_1\\ \alpha_3&-\alpha_2-2\rho_2&\alpha_1+2\rho_1&0,\end{pmatrix},
$$

where the α_is are such that A is compatible with the curvature matrix:

$$
F=\frac{k}{2}\begin{pmatrix}0&\Omega_1&\Omega_2&\Omega_3\\ -\Omega_1&0&-\Omega_3&\Omega_2\\ -\Omega_2&\Omega_3&0&-\Omega_1\\ -\Omega_3&-\Omega_2&\Omega_1&0\end{pmatrix},
$$

i.e. $F = dA + A \wedge A$. If $\{\xi_0, \xi_1, \xi_2, \xi_3\}$ are the vertical 1-forms dual to the basis induced by the σ_is on the vertical space, then, it is straightforward to check that the Spin(7)-structure:

$$\Phi = f^2 b_0 \wedge b_1 \wedge b_2 \wedge b_3 + g^2 \xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 + fg \sum_{i=1}^{3} \Omega_i \wedge (\xi_0 \wedge \xi_i - \xi_j \wedge \xi_k)$$

is torsion-free when f, g are functions only depending on the square of the distance from the zero section, r^2, and satisfying the ODEs in r^2:

$$\dot{(fg)} = \frac{k}{4} g^2, \quad \dot{(f^2)} = \frac{3k}{2} fg.$$

We call such examples deformed Spin(7) Bryant–Salamon manifolds. Indeed, if we pick $\alpha_i = -\rho_i$, then, E becomes the negative spinor bundle over B and we recover the Spin(7) Bryant–Salamon manifolds (cfr. Remark 2.3.14).

3.3.2 Isotropic linear Cayley fibrations

Under the same isotropy condition as in Section 3.2.2, we are able to solve the PDE system of Proposition 3.3.3. The proof is conceptually identical to the one for the G_2 case.

Theorem 3.3.5. *Let M be a Cayley vector bundle such that the horizontal and the vertical spaces are isotropic, i.e., for every $p \in M$ and unit $v, w \in H_p \subset T_pM$ (or $V_p \subset T_pM$) there exists a local isomorphism φ of (M, Φ) such that $\varphi_*(v) = w$. Then, (M, Φ) is locally isomorphic to a deformed* Spin(7) *Bryant–Salamon manifold.*

Proof. The isotropic condition implies that there are two functions, f, g on M such that $f^{1/2} = f_i$ and $g^{1/2} = g_i$ for all $i = 0, ..., 3$. Under this condition, the system Eq. (3.3.1) becomes:

$$\begin{cases} 0 = \sum_{l=0}^{3} \partial_{a_l}(f^2)\xi_l \wedge \mathrm{vol}_H + \sum_{i=1}^{3} fg\Omega_i \wedge \big((F^0_{\underline{a}} \wedge \xi_i - F^i_{\underline{a}} \wedge \xi_0) - (F^j_{\underline{a}} \wedge \xi_k - F^k_{\underline{a}} \wedge \xi_j)\big) \\ 0 = \sum_{l=0}^{3} \big(e_l(g^2) b_l - \partial_{a_l}(g^2) \sum_{m=0}^{3} A^l_m a_m\big) \\ 0 = \sum_{i=1}^{3} \big(-\sum_{l,m=0}^{3} \partial_{a_l}(fg) a_m A^l_m\big) \wedge (b_0 \wedge b_i - b_j \wedge b_k) \wedge (\xi_0 \wedge \xi_i - \xi_j \wedge \xi_k) \\ \qquad +\big(\sum_{l=0}^{3} e_l(fg) b_l\big) \wedge (b_0 \wedge b_i - b_j \wedge b_k) \wedge (\xi_0 \wedge \xi_i - \xi_j \wedge \xi_k) \\ \qquad +2fg(\rho_k + \tilde{\rho}_k) \wedge (b_0 \wedge b_j - b_k \wedge b_i) \wedge (\xi_0 \wedge \xi_i - \xi_j \wedge \xi_k) \\ \qquad -2fg(\rho_j + \tilde{\rho}_j) \wedge (b_0 \wedge b_k - b_i \wedge b_j) \wedge (\xi_0 \wedge \xi_i - \xi_j \wedge \xi_k) \\ 0 = \sum_{i=1}^{3} \sum_{l=0}^{3} \big(\partial_{a_l}(fg)\big)\xi_l \wedge (b_0 \wedge b_i - b_j \wedge b_k) \wedge (\xi_0 \wedge \xi_i - \xi_j \wedge \xi_k) \\ \quad +g^2 \sum_{l=0}^{3} (-1)^l (F^l_{\underline{a}} \wedge \xi_0 \wedge ... \wedge \hat{\xi}_l \wedge ... \wedge \xi_4) \end{cases}, \tag{3.3.2}$$

where $\Omega_i = b_0 \wedge b_i - b_j \wedge b_k$, $2\rho_i = \omega^k_j - \omega^i_0$ and $2\tilde{\rho}_i = A^i_0 - A^k_j$ for all (i, j, k) cyclic permutation of $(1, 2, 3)$.

We begin our analysis from the last equation, which, once expanded and regrouped term by term, can be rewritten as:

$$g^2 F^0_{\underline{a}} = \sum_{i=1}^{3} \partial_{a_i}(fg)\Omega_i, \tag{3.3.3}$$

$$g^2 F^i_{\underline{a}} = -\partial_{a_0}(fg)\Omega_i + \partial_{a_k}(fg)\Omega_j - \partial_{a_j}(fg)\Omega_k \quad \forall i = 1,2,3. \tag{3.3.4}$$

Since $\{F^i_j\}$ is antisymmetric, we also deduce that:

$$0 = -g^2 F^{a}_{\underline{a}} = \sum_{i=1}^{3} E_i(fg)\Omega_i,$$

where the E_is are the generators of $\mathfrak{sp}(1)$:

$$\begin{aligned} E_1 &= a_1\partial_{a_0} - a_0\partial_{a_1} - a_3\partial_{a_2} + a_2\partial_{a_3}, \\ E_2 &= a_2\partial_{a_0} + a_3\partial_{a_1} - a_0\partial_{a_2} - a_1\partial_{a_3}, \\ E_3 &= a_3\partial_{a_0} - a_2\partial_{a_1} + a_1\partial_{a_2} - a_0\partial_{a_3}. \end{aligned}$$

In particular, fg depends only on r^2 in the fibre, and hence, $\partial_{a_i}(fg) = 2a_i(\dot{fg})$. Using again Eq. (3.3.3) and Eq. (3.3.4), it is straightforward to obtain:

$$g^2 F^{E_i}_{\underline{a}} = 2(\dot{fg})r^2\Omega_i \tag{3.3.5}$$

for all $i = 1,2,3$, which also implies that $F^{E_1}_{\underline{a}01} = -F^{E_1}_{\underline{a}23} = F^{E_2}_{\underline{a}02} = \ldots = F^{E_3}_{\underline{a}12}$. Now, as $E_i(F^{E_i}_{\underline{a}0i}) = 0$, we conclude that $F^{E_i}_{\underline{a}}$ depends only on r^2 in the fibre. Therefore, this must be the case for g and f as well. Taking Eq. (3.3.5) along the coordinate lines we also obtain:

$$g^2\frac{k}{2} = 2(\dot{fg}), \tag{3.3.6}$$

where

$$k = 2F^0_{i0i} = -2F^0_{ijk} = 2F^i_{k0j} = -2F^i_{kki} = -2F^i_{j0k} = 2F^i_{jij}$$

for all (i,j,k) cyclic permutation of $(1,2,3)$.

We now turn our attention to the other equations. Plugging in Eq. (3.3.3) and Eq. (3.3.4) into the first equation of our system, we see that f and g need to satisfy:

$$g\dot{f} = 3(\dot{fg}). \tag{3.3.7}$$

The second equation of Eq. (3.3.2), implies that g does not depend on the basis. Therefore, we only have to study the third equation of Eq. (3.3.2).

Claim: Up to changing the metric on g_B, f can be assumed to be independent from the basis B.

This claim implies that k is a constant and that the third equation of the system becomes:

$$(\rho_k + \tilde{\rho}_k) \wedge (b_0 \wedge b_j - b_k \wedge b_i) - (\rho_j + \tilde{\rho}_j) \wedge (b_0 \wedge b_k - b_i \wedge b_j) = 0, \tag{3.3.8}$$

for all (i, j, k) positive permutation of $(1, 2, 3)$. Indeed, k depends only on the base by definition and only on r by Eq. (3.3.6).

It is straightforward to verify from Eq. (3.3.8) that $\rho_i + \tilde{\rho}_i = 0$, and hence, we can compute the curvature of the ASD two forms on B:

$$\begin{aligned} d(2\rho_i) + (2\rho_j \wedge 2\rho_k) &= -\left(d(2\tilde{\rho}_i) - (2\tilde{\rho}_j \wedge 2\tilde{\rho}_k)\right) \\ &= -F_0^i + F_j^k \\ &= k\Omega_i. \end{aligned}$$

We deduce that B must be self-dual and Einstein [19, Fact pag. 842] and we conclude.

Proof of the claim: As f, g are functions of r^2 in the fibres, the third equation of Eq. (3.3.2) becomes:

$$e_l(f) = 4fk_l, \tag{3.3.9}$$

where,

$$\begin{aligned} &k_0 := (\rho_i + \tilde{\rho}_i)_i \quad \forall i = 1, 2, 3, \\ &k_i := (\rho_j + \tilde{\rho}_j)_k = -(\rho_k + \tilde{\rho}_k)_j = -(\rho_i + \tilde{\rho}_i)_0 \quad \forall (i, j, k) \sim (1, 2, 3), \end{aligned}$$

and where we use the convention that, given an horizontal 1-form α, we denote by $(\alpha)_l$ the coefficient of α in the b_l term.

From Eq. (3.3.9), we deduce that $e_l(\log f)$ is independent from the fibres for every l, and hence, $f = f_B \cdot f_F$ where f_B is a function only depending on the basis and f_F is a function only depending on the fibre. The last crucial observation is that, changing the metric on B such that $\{f_B^{1/2} b_i\}_i$ form an orthonormal frame, we can reabsorb $e_l(f)$ into $4fk_l$ and assume f independent from the base. $\square$

$\square$

Remark 3.3.6. Observe that the isotropy condition rules out well-known examples coming from lower dimensional geometries. For instance, the product of a G_2 Bryant–Salamon manifold with a flat $\mathbb{R}$ is a Spin(7)-manifold of the form described in Proposition 3.3.3.

However, the f_i or g_i corresponding to the flat direction would need to be constant. This clearly gives a contradiction unless we are in the flat local model, where they all coincide. Another remarkable example is the one given by the Stenzel metric on T^*S^4. However, it is easy to see that this space doesn't satisfy the isotropic condition (see for instance equations (2.3), (2.4), (2.5) in [72]).

Chapter 4

Cayley fibrations in the Bryant–Salamon Spin(7) manifold

This chapter is devoted to the description of the author's construction of Cayley fibrations in the Bryant–Salamon Spin(7) manifolds (cfr. [71]). It is interesting to notice that the fibres of these fibrations provide new examples of Cayley submanifolds.

The main idea used in the construction is to reduce the problem to a system of ODEs via a group action. In particular, we consider G a 3-dimensional Lie group acting on the given Spin(7) manifold. By Theorem 2.3.8, there exists a unique Cayley passing through any 3-dimensional orbit of G. Hence, if the principal orbits are 3-dimensional, it is sensible to look for G-invariant Cayleys and fibrations.

Obviously, in the non-singular set, any G-invariant submanifold Σ can be seen as a 1-parameter family of G-orbits or, equivalently, as a curve in the space of orbits. Hence, the tangent space of Σ can be represented as the tangent space of the orbits together with the velocity vector field of the curve. One can now plug in these tangent vectors into τ (see Proposition 2.3.9) and obtain an explicit system of ODEs. The solutions of this system will parametrize the Cayley fibrations, as defined in Definition 3.1.5.

The obvious place where to look for such a group is the automorphism group, which is $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$ in our setting. In order to simplify our computations we will consider only the 3-dimensional subgroups that do not sit diagonally in $\mathrm{Sp}(2) \times \mathrm{Sp}(1)$. These were characterized in Section 2.3.3.3.

4.1 Cayley fibration invariant under the $\mathrm{Sp}(1)$-action on the fibre

Let $M := \not{S}_{-}(S^4)$ and $M_0 := \mathbb{R}^+ \times S^7$ endowed with the torsion-free Spin(7)-structures Φ_c constructed by Bryant and Salamon and described in Section 2.3.3.

Observe that (M, Φ_c) and (M_0, Φ_0) admit a trivial Cayley Fibration. Indeed, it is straightforward to see that the natural projection to S^4 realizes both spaces as honest Cayley fibrations with smooth fibres diffeomorphic to $\mathbb{R}^4$ and $\mathbb{R}^4 \setminus \{0\}$, respectively. In both cases, the parametrizing family is clearly S^4

The fibres are asymptotically conical to the cone of link S^3 and metric:

$$ds^2 + \frac{9}{25}s^2 g_{S^3},$$

where $s = r^{3/5}10/3$ and g_{S^3} is the standard unit round metric.

Since $\mathrm{Id}_{\mathrm{Sp}(2)} \times \mathrm{Sp}(1)$ acts trivially on the basis, and as $\mathrm{Sp}(1)$ on the fibres of $\not{S}_{-}(S^4)$ identified with $\mathbb{H}$, it is clear that the trivial fibration is invariant under $\mathrm{Id}_{\mathrm{Sp}(2)} \times \mathrm{Sp}(1)$.

Remark 4.1.1. We compute the associated multi-moment map, ν_c, as in Definition 2.4.6. This is:

$$\nu_c := \frac{20}{3}(r^2 - 5c)(c + r^2)^{1/5} + \frac{100}{3}c^{6/5},$$

where we subtracted $c^{6/5}100/3$ so that the range of the multi-moment map is $[0, \infty)$. Observe that the level sets of ν_c coincide with the level sets of the distance function from the zero section.

Remark 4.1.2. As in [49, Section 4.4], this fibration becomes the trivial Cayley fibration of $\mathbb{R}^8 = \mathbb{R}^4 \oplus \mathbb{R}^4$ when we blow-up at any point of the zero section.

4.2 Cayley fibration invariant under the lift of the $\mathrm{SO}(3) \times \mathrm{Id}_2$-action on S^4

Let $M := \not{S}_{-}(S^4)$ and $M_0 := \mathbb{R}^+ \times S^7$ be endowed with the torsion-free Spin(7)-structures Φ_c constructed by Bryant and Salamon that we described in Section 2.3.3. On each Spin(7) manifold, we construct the Cayley Fibration which is invariant under the lift to M (or M_0) of the standard $\mathrm{SO}(3) \times \mathrm{Id}_2$-action on $S^4 \subset \mathbb{R}^3 \oplus \mathbb{R}^2$.

4.2.1 Choice of coframe on S^4

As in [49], we choose an adapted orthonormal coframe on S^4 which is compatible with the symmetries we will impose. Since the action coincides, when restricted to S^4, with the one used by Karigiannis and Lotay on $\Lambda^2_-(T^*S^4)$ [49, Section 5], it is natural to employ the same coframe, which we now recall.

We split $\mathbb{R}^5$ into the direct sum of a 3-dimensional vector subspace $P \cong \mathbb{R}^3$ and its orthogonal complement $P^\perp \cong \mathbb{R}^2$. As S^4 is the unit sphere in $\mathbb{R}^5$, we can write, with respect to this splitting:

$$S^4 = \left\{(\mathbf{x}, \mathbf{y}) \in P \oplus P^\perp : |\mathbf{x}|^2 + |\mathbf{y}|^2 = 1\right\}.$$

Now, for all $(\mathbf{x}, \mathbf{y}) \in S^4$ there exists a unique $\alpha \in [0, \pi/2]$, some $\mathbf{u} \in S^2 \subset P$ and some $\mathbf{v} \in S^1 \subset P^\perp$ such that:

$$\mathbf{x} = \cos\alpha\mathbf{u}, \qquad \mathbf{y} = \sin\alpha\mathbf{v}.$$

Observe that $\mathbf{u}$ and $\mathbf{v}$ are uniquely determined when $\alpha \in (0, \pi/2)$, while, when $\alpha = 0, \pi/2$, $\mathbf{v}$ can be any unit vector in $P^\perp$ ($\mathbf{y} = 0$) and $\mathbf{u}$ can be any unit vector in P ($\mathbf{x} = 0$), respectively. Hence, we are writing S^4 as the disjoint union of an S^2, corresponding to $\alpha = 0$, of an S^1, corresponding to $\alpha = \pi/2$, and of $S^2 \times S^1 \times (0, \pi/2)$.

If we put spherical coordinates on S^2 and polar coordinates on S^1, then, we can write

$$\mathbf{u} = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi),$$

and

$$\mathbf{v} = (\cos\beta, \sin\beta),$$

where $\theta \in [0, \pi]$, $\phi \in [0, 2\pi)$ and $\beta \in [0, 2\pi)$. As usual, ϕ is not unique when $\theta = 0, \pi$.

It follows that, if we take out the points where $\theta = 0, \pi$ from $S^2 \times S^1 \times (0, \pi/2)$, we have constructed a coordinate patch U parametrized by $(\alpha, \beta, \theta, \phi)$ on S^4. Explicitly, U is S^4 minus two totally geodesic S^2:

$$S^2_{y_1, y_2 = 0} = \left\{(\mathbf{x}, \mathbf{0}) \in P \oplus P^\perp : |x|^2 = 1\right\},$$

corresponding to $\alpha = 0$, and

$$S^2_{x_2, x_3 = 0} = \left\{(\cos\alpha, 0, 0, \sin\alpha\cos\beta, \sin\alpha\sin\beta) \in P \oplus P^\perp : \alpha \in (0, \pi)\right\},$$

corresponding to $\theta = 0$ and $\theta = \pi$. Observe, that the S^1 corresponding to $\alpha = \pi/2$ is a totally geodesic equator in $S^2_{x_2, x_3 = 0}$.

A straightforward computation shows that the coordinate frame $\{\partial_\alpha, \partial_\beta, \partial_\theta, \partial_\phi\}$ is orthogonal and can be easily normalized obtaining:

$$f_0 := \partial_\alpha, \quad f_1 := \frac{\partial_\beta}{\sin\alpha}, \quad f_2 := \frac{\partial_\theta}{\cos\alpha}, \quad f_3 := \frac{\partial_\phi}{\cos\alpha\sin\theta}.$$

The dual orthonormal coframe is given by:

$$b_0 := d\alpha, \quad b_1 := \sin\alpha d\beta, \quad b_2 := \cos\alpha d\theta, \quad b_3 := \cos\alpha\sin\theta d\phi. \tag{4.2.1}$$

Observe that $\{b_0, b_1, b_2, b_3\}$ is positively oriented with respect to the outward pointing normal of S^4, hence, the volume form is:

$$\mathrm{vol}_{S^4} = \sin\alpha\cos^2\alpha\sin\theta d\alpha\wedge d\beta\wedge d\theta\wedge d\phi.$$

4.2.2 Horizontal and the vertical space

As in [49, Subsection 5.2], we use Equation (2.3.1) to compute the ρ_i's in the coordinate frame we have just defined. Indeed, Eq. (4.2.1) implies that:

$$\begin{aligned}
\Omega_1 &= \sin\alpha d\alpha\wedge d\beta - \cos^2\alpha\sin\theta d\theta\wedge d\phi,\\
\Omega_2 &= \cos\alpha d\alpha\wedge d\theta - \sin\alpha\cos\alpha\sin\theta d\phi\wedge d\beta,\\
\Omega_3 &= \cos\alpha\sin\theta d\alpha\wedge d\phi - \sin\alpha\cos\alpha d\beta\wedge d\theta;
\end{aligned} \tag{4.2.2}$$

hence, we deduce that:

$$\begin{aligned}
d\Omega_1 &= 2\sin\alpha\cos\alpha\sin\theta d\alpha\wedge d\theta\wedge d\phi,\\
d\Omega_2 &= (\sin^2\alpha - \cos^2\alpha)\sin\theta d\alpha\wedge d\phi\wedge d\beta - \sin\alpha\cos\alpha\cos\theta d\theta\wedge d\phi\wedge d\beta,\\
d\Omega_3 &= \cos\alpha\cos\theta d\theta\wedge d\alpha\wedge d\phi + (\sin^2\alpha - \cos^2\alpha) d\alpha\wedge d\beta\wedge d\theta.
\end{aligned}$$

We conclude that in these coordinates we have:

$$2\rho_1 = -\cos\alpha d\beta + \cos\theta d\phi; \quad 2\rho_2 = \sin\alpha d\theta; \quad 2\rho_3 = \sin\alpha\sin\theta d\phi.$$

Now that we have computed the connection forms, we immediately see from Eq. (2.3.2) that the vertical one forms are:

$$\begin{aligned}
\xi_0 &= da_0 + a_1\left(-\frac{\cos\alpha}{2}d\beta + \frac{\cos\theta}{2}d\phi\right) + a_2\frac{\sin\alpha}{2}d\theta + a_3\frac{\sin\alpha\sin\theta}{2}d\phi,\\
\xi_1 &= da_1 - a_0\left(-\frac{\cos\alpha}{2}d\beta + \frac{\cos\theta}{2}d\phi\right) - a_2\frac{\sin\alpha\sin\theta}{2}d\phi + a_3\frac{\sin\alpha}{2}d\theta,\\
\xi_2 &= da_2 - a_0\frac{\sin\alpha}{2}d\theta + a_1\frac{\sin\alpha\sin\theta}{2}d\phi - a_3\left(-\frac{\cos\alpha}{2}d\beta + \frac{\cos\theta}{2}d\phi\right),\\
\xi_3 &= da_3 - a_0\frac{\sin\alpha\sin\theta}{2}d\phi - a_1\frac{\sin\alpha}{2}d\theta + a_2\left(-\frac{\cos\alpha}{2}d\beta + \frac{\cos\theta}{2}d\phi\right).
\end{aligned} \tag{4.2.3}$$

4.2.3 SU(2)-action

Given the splitting of Section 4.2.1, $\mathbb{R}^5 = P \oplus P^\perp$, since $P \cong \mathbb{R}^3$ and $P^\perp \cong \mathbb{R}^2$, we can consider SO(3) acting in the usual way on P and trivially on $P^\perp$. In other words, we see $\mathrm{SO}(3) \cong \mathrm{SO}(P) \times \mathrm{Id}_{P^\perp} \subset \mathrm{SO}(P \oplus P^\perp) \cong \mathrm{SO}(5)$. Obviously, this is also an action on S^4.

By taking the differential, SO(3) acts on the frame bundle $P_{\mathrm{SO}(4)}$ of S^4. The theory of covering spaces implies that this action lifts to a $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$-action on the spin structure $P_{\mathrm{Spin}(4)}$ of S^4. In particular, the following diagram is commutative:

$$\begin{tikzcd} & & P_{\mathrm{Spin}(4)} \arrow[d, "\tilde{\pi}"] \\ \mathrm{Spin}(3) \times P_{\mathrm{Spin}(4)} \arrow[r, "\tilde{\pi}_0^3 \times \tilde{\pi}"'] \arrow[urr, dotted] & \mathrm{SO}(3) \times P_{\mathrm{SO}(4)} \arrow[r] & P_{\mathrm{SO}(4)} \end{tikzcd} . \tag{4.2.4}$$

Finally, if Spin(3) acts trivially on $\mathbb{H}$, we can combine the two Spin(3)-actions to obtain one on $P_{\mathrm{Spin}(4)} \times \mathbb{H}$, which descends to the quotient $P_{\mathrm{Spin}(4)} \times_{\mu_-} \mathbb{H} = \slashed{S}_-(S^4)$.

Remark 4.2.1. Recall that $TS^4 = P_{\mathrm{SO}(4)} \times_{\cdot} \mathbb{R}^4$, where $\cdot$ is the standard representation of SO(4) on $\mathbb{R}^4$. Let G be a subgroup of SO(5) which acts on $P_{\mathrm{SO}(4)} \times_{\cdot} \mathbb{R}^4$ via the differential on the first term and trivially on the second. It is straightforward to verify that this action passes to the quotient and that it coincides with the differential on TS^4.

Now, we describe the geometry of this Spin(3)-action on $\slashed{S}_-(S^4)$. Since $\tilde{\pi}$ is fibre-preserving and (Eq. (4.2.4)) represents a commutative diagram, we observe that, fixed a point $p = (\mathbf{x}, \mathbf{y}) \in S^4 \subset P \oplus P^\perp$, the subgroup of Spin(3) that preserves the fibre of $P_{\mathrm{Spin}(4)}$ over p is the lift of the subgroup of SO(3) that fixes the fibre of $P_{\mathrm{SO}(4)}$ over p.

We first assume $\alpha \neq \pi/2$. The subgroup of SO(3) that preserves the fibres of $P_{\mathrm{SO}(4)}$ rotates the tangent space of $S^2 \subset P$ and fixes the other vectors tangent to S^4. Explicitly, if $\{e_i\}_{i=0}^3$ is the oriented orthonormal frame of Section 4.2.1 (or an analogous frame when $\alpha = 0$, $\theta = 0, \pi$), the transformation matrix under the action is:

$$h_\gamma := \left[\begin{array}{c|cc} \mathrm{Id}_2 & & \\ \hline & \cos\gamma & -\sin\gamma \\ & \sin\gamma & \cos\gamma \end{array}\right] \in \mathrm{SO}(4), \tag{4.2.5}$$

for some $\gamma \in [0, 2\pi)$.

Claim 1. *For all* $\gamma \in [0, 4\pi)$*, under the isomorphism* $\mathrm{Spin}(4) \cong \mathrm{Sp}(1) \times \mathrm{Sp}(1)$*, we have:*

$$\tilde{\pi}_0^4(\tilde{h}_\gamma) = h_\gamma,$$

where $\tilde{h}_\gamma = (\cos(\gamma/2) + i\sin(\gamma/2), \cos(\gamma/2) + i\sin(\gamma/2))$.

Proof. It is well-known that, in this context, $\tilde{\pi}_0^4\,((l,r))\cdot a = la\bar{r}$ for all $(l,r) \in \mathrm{Sp}(1)\times\mathrm{Sp}(1)$ and all $a \in \mathbb{H} \cong \mathbb{R}^4$. The claim follows from a straightforward computation. $\square$

Using once again the commutativity of (Eq. (4.2.4)) and Claim 1, we deduce that the action in the trivialization of $\not{S}_-(S^4)$ induced by $\{e_i\}_{i=0}^3$ is as follows:

$$\begin{array}{ccccccc} U\times\mathbb{H} & \xrightarrow{\cong} & (U\times\mathrm{Spin}(4))\times_{\mu_-}\mathbb{H} & \longrightarrow & (U\times\mathrm{Spin}(4))\times_{\mu_-}\mathbb{H} & \xrightarrow{\cong} & U\times\mathbb{H} \\ (p,a) & \longmapsto & \left[(p,1_{\mathrm{Spin}(4)}),a\right] & \longmapsto & \left[(p,\tilde{h}_\gamma),a\right] & \longmapsto & \left(p,a\hat{h}_\gamma\right) \end{array},$$

where $\hat{h}_\gamma := \cos(\gamma/2) - i\sin(\gamma/2)$ and where $a \in \mathbb{H}$. If we write both $\mathbb{R}^2$ factors of $\mathbb{H} \cong \mathbb{R}^2 \oplus \mathbb{R}^2$ in polar coordinates, i.e.,

$$a = s\cos(\gamma_-/2) + is\sin(\gamma_-/2) + jt\cos(\gamma_+/2) + kt\sin(\gamma_+/2),$$

for $s,t \in [0,\infty)$ and $\gamma_\pm \in [0,4\pi)$, we observe that

$$a\hat{h}_\gamma = s\cos\left((\gamma_- - \gamma)/2\right) + is\sin\left((\gamma_- - \gamma)/2\right) + jt\cos\left((\gamma_+ + \gamma)/2\right) + kt\sin((\gamma_+ + \gamma)/2).$$

Geometrically, this is a rotation of angle $-\gamma/2$ on the first $\mathbb{R}^2$ and of angle $\gamma/2$ on the second.

Now, we assume $\alpha = \pi/2$. In this case, the whole $\mathrm{Spin}(3)$ fixes the fibre of $\not{S}_-(S^4)$.

Claim 2. $\mathrm{Spin}(3)$ *acts on the fibre of* $\not{S}_-(S^4)$ *as* $\mathrm{Sp}(1)$ *acts on* $\mathbb{H}$ *via right multiplication of the quaternionic conjugate.*

Proof. Consider an orthonormal frame such that, at $p = (\underline{0}, \cos\beta, \sin\beta)$, has the form:

$$e_0 = -\sin\beta\partial_3 + \cos\beta\partial_4; \qquad e_1 = \partial_0; \qquad e_2 = \partial_1; \qquad e_3 = \partial_2,$$

where ∂_i are the coordinate vectors of $\mathbb{R}^5 \cong P \oplus P^\perp$. Observe that the SO(3)-action fixes e_0 and acts on e_1, e_2, e_3 via matrix multiplication. In particular, given $G \in \mathrm{SO}(3)$, the transformation matrix of the frame at p is:

$$\left[\begin{array}{c|c} 1 & \\ \hline & G \end{array}\right].$$

Moreover, for all $g \in \mathrm{Sp}(1) \cong \mathrm{Spin}(3)$ and $(g,g) \in \mathrm{Sp}(1)\times\mathrm{Sp}(1) \cong \mathrm{Spin}(4)$, then

$$\tilde{\pi}_0^4((g,g)) = \left[\begin{array}{c|c} 1 & \\ \hline & \tilde{\pi}_0^3(g) \end{array}\right],$$

where we recall that $\tilde{\pi}_0^3(l) \cdot x = lx\bar{l}$ for all $l \in \mathrm{Sp}(1)$ and $x \in \mathrm{Im}\mathbb{H} \cong \mathbb{R}^3$. Indeed, the left-hand side reads:

$$\tilde{\pi}_0^4((g, g)) \cdot a = ga\overline{g} = g\mathrm{Re}(a)\overline{g} + g\mathrm{Im}(a)\overline{g} = \mathrm{Re}(a) + g\mathrm{Im}(a)\overline{g},$$

while the right-hand side is:

$$\left[\begin{array}{c|c} 1 & \\ \hline & \tilde{\pi}_0^3(g) \end{array}\right] a = \begin{pmatrix} \mathrm{Re}(a) \\ g\mathrm{Im}(a)\overline{g} \end{pmatrix}.$$

We conclude the proof through the commutativity of (Eq. (4.2.4)). □

We put all these observations in a lemma.

Lemma 4.2.2. *The orbits of the* SU(2) $\cong$ Spin(3)*-action on* $\not{S}_-(S^4)$ *are given in Table Table 4.1.*

α	(s,t)	Orbit
$\neq \frac{\pi}{2}$	$= (0,0)$	S^2
$\neq \frac{\pi}{2}$	$\neq (0,0)$	S^3
$= \frac{\pi}{2}$	$= (0,0)$	Point
$= \frac{\pi}{2}$	$\neq (0,0)$	S^3

Table 4.1: Spin(3) Orbits

4.2.4 SU(2) adapted coordinates

The description of the SU(2)-action that we carried out in Section 4.2.3 suggests the following reparametrization of the linear coordinates (a_0, a_1, a_2, a_3) on the fibres of $\not{S}_-(S^4)$:

$$a_0 = s\cos\left(\frac{\delta-\gamma}{2}\right); \quad a_1 = s\sin\left(\frac{\delta-\gamma}{2}\right); \quad a_2 = t\cos\left(\frac{\delta+\gamma}{2}\right); \quad a_3 = t\sin\left(\frac{\delta+\gamma}{2}\right), \tag{4.2.6}$$

where $s, t \in [0, \infty)$, $\gamma \in [0, 4\pi)$ and $\delta \in [0, 2\pi)$. This is a well-defined coordinate system when s and t are strictly positive; we will assume this from now on. Geometrically, γ represents the SU(2)-action, while δ can be either seen as the phase in the orbit of the action when $(a_0, a_1) = (s, 0)$ or as twice the common angle in $[0, \pi)$ that the suitable point in the orbit makes with $(s, 0)$ and $(t, 0)$. These interpretations can be recovered by putting $\gamma = \delta$ and $\gamma = 0$, respectively.

Similarly to [49], we introduce the standard left-invariant coframe on SU(2) of coordinates γ, θ, ϕ defined on the same intervals as above:

$$\sigma_1 = d\gamma + \cos\theta d\phi; \quad \sigma_2 = \cos\gamma d\theta + \sin\gamma\sin\theta d\phi; \quad \sigma_3 = \sin\gamma d\theta - \cos\gamma\sin\theta d\phi. \tag{4.2.7}$$

Observe that:

$$\sigma_2 \wedge \sigma_3 = -\sin\theta d\theta \wedge d\phi. \tag{4.2.8}$$

Our choice of parametrization of $\slashed{S}_-(S^4)$ implies that Eq. (4.2.7) is a coframe on the 3-dimensional orbits of the SU(2)-action.

So far, we have constructed a coordinate system $\alpha, \beta, \theta, \phi, s, t, \delta, \gamma$ defining a chart $\mathcal{U}$ of $\slashed{S}_-(S^4)$ and a coframe $\{\sigma_1, \sigma_2, \sigma_3, d\alpha, d\beta, ds, dt, d\delta\}$ on that chart. These coordinates and coframe are such that γ, θ, ϕ parametrize the orbits of the SU(2)-action and $\{\sigma_1, \sigma_2, \sigma_3\}$ forms a coframe on these orbits. Let $\{\partial_1, \partial_2, \partial_3, \partial_\alpha, \partial_\beta, \partial_s, \partial_t, \partial_\delta\}$ be the relative dual frame.

4.2.5 Spin(7) geometry in the adapted coordinates

In this subsection, we write the Cayley form Φ_c, as in Eq. (2.3.3), and the relative metric g_c, as in Eq. (2.3.4), with respect to the SU(2) adapted coordinates defined in Section 4.2.4.

Lemma 4.2.3. *The horizontal 2-forms Ω_1, Ω_2, Ω_3, in the adapted frame defined in Section 4.2.4, satisfy:*

$$\Omega_1 = \sin\alpha d\alpha \wedge d\beta + \cos^2\alpha \sigma_2 \wedge \sigma_3$$

and

$$\begin{aligned} \cos\gamma\Omega_2 + \sin\gamma\Omega_3 &= \cos\alpha(d\alpha \wedge \sigma_2 - \sin\alpha d\beta \wedge \sigma_3), \\ -\sin\gamma\Omega_2 + \cos\gamma\Omega_3 &= \cos\alpha(-d\alpha \wedge \sigma_3 - \sin\alpha d\beta \wedge \sigma_2). \end{aligned}$$

Proof. The equations follow from Eq. (4.2.2), Eq. (4.2.7) and Eq. (4.2.8). □

Lemma 4.2.4. *The vertical 2-forms A_1, A_2, A_3, in the adapted frame defined in Section 4.2.4, have the form:*

$$\begin{aligned} A_1 =& \frac{1}{2}(sds - tdt) \wedge d\delta + \frac{\cos\alpha}{2}(sds + tdt) \wedge d\beta - \frac{1}{2}(sds + tdt) \wedge \sigma_1 \\ &+ \frac{\sin\alpha}{2}(tds - sdt) \wedge \sigma_3 + (s^2 + t^2)\frac{\sin^2\alpha}{4}\sigma_2 \wedge \sigma_3 + \frac{st\sin\alpha}{2}\sigma_2 \wedge d\delta, \end{aligned}$$

$$
\begin{aligned}
A_2 =& \cos\gamma ds\wedge dt - \frac{t}{2}\sin\gamma ds\wedge(d\gamma+d\delta) - \frac{s}{2}\sin\gamma dt\wedge(d\delta-d\gamma) - \frac{st}{2}\cos\gamma d\gamma\wedge d\delta\\
&- (s^2+t^2)\frac{\sin\alpha\cos\alpha}{4}\sin\theta d\beta\wedge d\phi + \sin\gamma(sdt-tds)\wedge\left(-\frac{\cos\alpha}{2}d\beta+\frac{\cos\theta}{2}d\phi\right)\\
&+ st\cos\gamma d\delta\wedge\left(-\frac{\cos\alpha}{2}d\beta+\frac{\cos\theta}{2}d\phi\right) + \frac{\sin\alpha}{2}d\theta\wedge(tdt+sds)\\
&+ \frac{t^2\sin\alpha\sin\theta}{4}(d\gamma+d\delta)\wedge d\phi + \frac{s^2\sin\alpha\sin\theta}{4}d\phi\wedge(d\delta-d\gamma);\\
A_3 =& \sin\gamma ds\wedge dt + \frac{t}{2}\cos\gamma ds\wedge(d\gamma+d\delta) + \frac{s}{2}\cos\gamma dt\wedge(d\delta-d\gamma) + \frac{st}{2}\sin\gamma d\delta\wedge d\gamma\\
&+ (s^2+t^2)\frac{\sin\alpha}{4}(\cos\alpha d\beta\wedge d\theta + \cos\theta d\theta\wedge d\phi)\\
&- \cos\gamma(sdt-tds)\wedge\left(-\frac{\cos\alpha}{2}d\beta+\frac{\cos\theta}{2}d\phi\right)\\
&+ st\sin\gamma d\delta\wedge\left(-\frac{\cos\alpha}{2}d\beta+\frac{\cos\theta}{2}d\phi\right) + \frac{\sin\alpha\sin\theta}{2}d\phi\wedge(tdt+sds)\\
&+ \frac{t^2\sin\alpha}{4}d\theta\wedge(d\gamma+d\delta) + \frac{s^2\sin\alpha}{4}(d\delta-d\gamma)\wedge d\theta.
\end{aligned}
\tag{4.2.9}
$$

Proof. Computing the exterior derivatives of the a_i's in the coordinates of Eq. (4.2.6), we can reduce our statement to a long computation based on Eq. (4.2.3). $\square$

Corollary 4.2.5. *The vertical 2-forms A_1, A_2, A_3, in the adapted frame defined in Section 4.2.4, satisfy:*

$$
\begin{aligned}
A_1 =& \left(ds+\frac{t\sin\alpha}{2}\sigma_2\right)\wedge\left(\frac{s}{2}d\delta+\frac{s\cos\alpha}{2}d\beta-\frac{s}{2}\sigma_1+\frac{t\sin\alpha}{2}\sigma_3\right)\\
&-\left(dt-\frac{s\sin\alpha}{2}\sigma_2\right)\wedge\left(\frac{t}{2}d\delta-\frac{t\cos\alpha}{2}d\beta+\frac{t}{2}\sigma_1+\frac{s\sin\alpha}{2}\sigma_3\right)
\end{aligned}
\tag{4.2.10}
$$

and

$$
\begin{aligned}
\cos\gamma A_2+\sin\gamma A_3 =& \left(ds+\frac{t\sin\alpha}{2}\sigma_2\right)\wedge\left(dt-\frac{s\sin\alpha}{2}\sigma_2\right)\\
&+\left(\frac{s}{2}d\delta+\frac{s\cos\alpha}{2}d\beta-\frac{s}{2}\sigma_1+\frac{t\sin\alpha}{2}\sigma_3\right)\wedge\\
&\wedge\left(\frac{t}{2}d\delta-\frac{t\cos\alpha}{2}d\beta+\frac{t}{2}\sigma_1+\frac{s\sin\alpha}{2}\sigma_3\right);
\end{aligned}
\tag{4.2.11}
$$

$$
\begin{aligned}
\cos\gamma A_3-\sin\gamma A_2 =& \left(ds+\frac{t\sin\alpha}{2}\sigma_2\right)\wedge\left(\frac{t}{2}d\delta-\frac{t\cos\alpha}{2}d\beta+\frac{t}{2}\sigma_1+\frac{s\sin\alpha}{2}\sigma_3\right)\\
&+\left(dt-\frac{s\sin\alpha}{2}\sigma_2\right)\wedge\left(\frac{s}{2}d\delta+\frac{s\cos\alpha}{2}d\beta-\frac{s}{2}\sigma_1+\frac{t\sin\alpha}{2}\sigma_3\right).
\end{aligned}
\tag{4.2.12}
$$

Proof. The first equation in Lemma 4.2.4 is exactly the development of Eq. (4.2.10).

A straightforward computation, involving Eq. (4.2.9), gives:

$$\begin{aligned}
\cos\gamma A_2 + \sin\gamma A_3 = {} & ds\wedge dt + \frac{st}{2} d\delta\wedge\sigma_1 - \frac{st}{2}\cos\alpha d\delta\wedge d\beta + (s^2+t^2)\frac{\sin\alpha\cos\alpha}{4} d\beta\wedge\sigma_3 \\
& + \frac{\sin\alpha}{2}\sigma_2\wedge(tdt+sds) + \frac{(t^2-s^2)\sin\alpha}{4}\sigma_3\wedge d\delta \\
& - \frac{(t^2+s^2)\sin\alpha}{4}\sigma_1\wedge\sigma_3; \\
\cos\gamma A_3 - \sin\gamma A_2 = {} & \frac{1}{2}(tds - sdt)\wedge\sigma_1 + \frac{1}{2}(tds+sdt)\wedge d\delta + (s^2+t^2)\frac{\sin\alpha\cos\alpha}{4} d\beta\wedge\sigma_2 \\
& + \frac{\cos\alpha}{2}(sdt - tds)\wedge d\beta + \frac{\sin\alpha}{2}(tdt+sds)\wedge\sigma_3 \\
& - \frac{(s^2+t^2)\sin\alpha}{4}\sigma_1\wedge\sigma_2 + \frac{(t^2-s^2)\sin\alpha}{4}\sigma_2\wedge d\delta;
\end{aligned}$$

which coincide with the development of Eq. (4.2.11) and Eq. (4.2.12), respectively. $\square$

Remark 4.2.6. Using the identities:

$$\begin{aligned}
b_0\wedge b_1\wedge b_2\wedge b_3 &= -\frac{1}{2}\Omega_1\wedge\Omega_1, \\
\xi_0\wedge\xi_1\wedge\xi_2\wedge\xi_3 &= -\frac{1}{2}A_1\wedge A_1
\end{aligned} \tag{4.2.13}$$

and

$$\begin{aligned}
\sum_{i=1}^{3} A_i\wedge\Omega_i = {} & A_1\wedge\Omega_1 + (\cos\gamma\Omega_2 + \sin\gamma\Omega_3)\wedge(\cos\gamma A_2 + \sin\gamma A_3) \\
& + (-\sin\gamma\Omega_2 + \cos\gamma\Omega_3)\wedge(-\sin\gamma A_2 + \cos\gamma A_3),
\end{aligned} \tag{4.2.14}$$

one could easily find Φ_c in the adapted frame of Section 4.2.4. It is clear from Corollary 4.2.5 that it is not going to be in a nice form.

Lemma 4.2.7. *Given $c \geq 0$, the Riemannian metric g_c, in the adapted frame of Section 4.2.4, takes the form:*

$$\begin{aligned}
g_c = {} & 5(c+r^2)^{3/5}\left(d\alpha^2 + \sin^2\alpha d\beta^2 + \cos^2\alpha(\sigma_2^2+\sigma_3^2)\right) \\
& + 4(c+r^2)^{-2/5}\Bigg(ds^2 + dt^2 + \frac{r^2\cos^2\alpha}{4}d\beta^2 + \frac{r^2}{4}\sigma_1^2 - \frac{r^2\cos\alpha}{2}d\beta\sigma_1 + \frac{r^2\sin^2\alpha}{4}(\sigma_2^2+\sigma_3^2) \\
& + \frac{(t^2-s^2)}{2}d\delta\sigma_1 + (st\sin\alpha)d\delta\sigma_3 + \frac{r^2}{4}d\delta^2 + \sin\alpha(tds - sdt)\sigma_2 - \frac{(t^2-s^2)\cos\alpha}{2}d\delta d\beta\Bigg),
\end{aligned}$$

where $r^2 = s^2 + t^2$.

Proof. Combining Eq. (2.3.4), Eq. (4.2.1), Eq. (4.2.3) and Eq. (4.2.7)), it is easy to obtain the Riemannian metric in the claimed form. $\square$

4.2.6 Diagonalizing coframe and frame

In this subsection we define the last coframe on $\$_{-}(S^4)$ that we will use. The motivation comes from the form of A_1, $\cos\gamma A_2 + \sin\gamma A_3$ and $\cos\gamma A_3 - \sin\gamma A_2$ that we obtained in Eq. (4.2.10), Eq. (4.2.11) and Eq. (4.2.12), respectively. We let:

$$
\begin{aligned}
\tilde{ds} &= ds + \frac{t\sin\alpha}{2}\sigma_2; & \tilde{dt} &= dt - \frac{s\sin\alpha}{2}\sigma_2;\\
\omega_1 &= sd\delta + s\cos\alpha d\beta - s\sigma_1 + t\sin\alpha\sigma_3; & \omega_2 &= td\delta - t\cos\alpha d\beta + t\sigma_1 + s\sin\alpha\sigma_3.
\end{aligned}
\tag{4.2.15}
$$

Since $t\omega_1 + s\omega_2 = 2tsd\delta + (t^2+s^2)\sin\alpha\sigma_3$ and $s\omega_2 - t\omega_1 = 2st\sigma_1 - 2st\cos\alpha d\beta + (s^2-t^2)\sin\alpha\sigma_3$, it is clear that $\{\sigma_2, \sigma_3, d\alpha, d\beta, \omega_1, \omega_2, \tilde{ds}, \tilde{dt}\}$ is a coframe on $\mathcal{U}$. Let $\{e_2, e_3, e_\alpha, e_\beta, e_{\omega_1}, e_{\omega_2}, e_s, e_t\}$ denote the relative dual frame.

Corollary 4.2.8. *The vertical 2-forms A_1, A_2, A_3, in the coframe defined in this subsection, satisfy:*

$$
A_1 = \frac{1}{2}\left(\tilde{ds}\wedge\omega_1 - \tilde{dt}\wedge\omega_2\right) \tag{4.2.16}
$$

and

$$
\cos\gamma A_2 + \sin\gamma A_3 = \tilde{ds}\wedge\tilde{dt} + \frac{1}{4}\omega_1\wedge\omega_2; \tag{4.2.17}
$$

$$
\cos\gamma A_3 - \sin\gamma A_2 = \frac{1}{2}\left(\tilde{ds}\wedge\omega_2 + \tilde{dt}\wedge\omega_1\right). \tag{4.2.18}
$$

Proof. It follows immediately from Corollary 4.2.5 and Eq. (4.2.15). □

Proposition 4.2.9. *Given $c \geq 0$, the Cayley form Φ_c, in the coframe defined in this subsection, satisfies:*

$$
\begin{aligned}
\Phi_c =& 4(c+r^2)^{-4/5}\tilde{ds}\wedge\tilde{dt}\wedge\omega_2\wedge\omega_1 + 25(c+r^2)^{6/5}\sin\alpha\cos^2\alpha d\alpha\wedge d\beta\wedge\sigma_3\wedge\sigma_2\\
& 10(c+r^2)^{1/5}\Big(\left(\tilde{ds}\wedge\omega_1 - \tilde{dt}\wedge\omega_2\right)\wedge\left(\sin\alpha d\alpha\wedge d\beta + \cos^2\alpha\sigma_2\wedge\sigma_3\right)\\
& + \frac{1}{2}\left(4\tilde{ds}\wedge\tilde{dt} + \omega_1\wedge\omega_2\right)\wedge(\cos\alpha(d\alpha\wedge\sigma_2 - \sin\alpha d\beta\wedge\sigma_3))\\
& + \left(\tilde{ds}\wedge\omega_2 + \tilde{dt}\wedge\omega_1\right)\wedge\cos\alpha\left(-d\alpha\wedge\sigma_3 - \sin\alpha d\beta\wedge\sigma_2\right)\Big),
\end{aligned}
\tag{4.2.19}
$$

where $r^2 = s^2 + t^2$.

Proof. This is a straightforward consequence of Lemma 4.2.3, Eq. (4.2.13), Eq. (4.2.14) and Corollary 4.2.8. □

Proposition 4.2.10. *Given $c \geq 0$, the Riemannian metric g_c, in the coframe defined in this subsection, satisfies:*

$$\begin{aligned} g_c =& 5(c+r^2)^{3/5}\left(d\alpha^2 + \sin^2\alpha d\beta^2 + \cos^2\alpha\left(\sigma_2^2+\sigma_3^2\right)\right) \\ &+ 4(c+r^2)^{-2/5}\left(\tilde{ds}^2 + \tilde{dt}^2 + \frac{(\omega_1^2+\omega_2^2)}{4}\right), \end{aligned} \tag{4.2.20}$$

where $r^2 = s^2 + t^2$.

Proof. The first addendum remains invariant from Lemma 4.2.7, while Eq. (4.2.15) implies that the remaining part is equal to the second addendum in Lemma Lemma 4.2.7. □

In particular, using this coframe, we sacrifice compatibility with the group action to obtain a simpler form for Φ_c and a diagonal metric.

We conclude this subsection by computing the dual frame with respect to the SU(2) adapted frame $\{\partial_1, \partial_2, \partial_3, \partial_\alpha, \partial_\beta, \partial_s, \partial_t, \partial_\delta\}$.

Lemma 4.2.11. *The dual frame $\{e_2, e_3, e_\alpha, e_\beta, e_{\omega_1}, e_{\omega_2}, e_s, e_t\}$ satisfies:*

$$\begin{aligned} e_\alpha &= \partial_\alpha; & e_\beta &= \partial_\beta + \cos\alpha\partial_1; \\ e_2 &= \partial_2 - \frac{t\sin\alpha}{2}\partial_s + \frac{s\sin\alpha}{2}\partial_t; & e_3 &= \partial_3 - \frac{(s^2+t^2)\sin\alpha}{2st}\partial_\delta + \frac{(t^2-s^2)\sin\alpha}{2st}\partial_1; \\ e_s &= \partial_s; & e_t &= \partial_t; \\ e_{\omega_1} &= \frac{1}{2s}\partial_\delta - \frac{1}{2s}\partial_1; & e_{\omega_2} &= \frac{1}{2t}\partial_\delta + \frac{1}{2t}\partial_1; \end{aligned} \tag{4.2.21}$$

where $\{\partial_1, \partial_2, \partial_3, \partial_\alpha, \partial_\beta, \partial_s, \partial_t, \partial_\delta\}$ is the dual frame with respect to the SU(2) *adapted coordinates of Section 4.2.4.*

Proof. It is straightforward to verify these identities from Eq. (4.2.15) and the definition of dual frame. □

4.2.7 Cayley condition

As the generic orbit of the SU(2)-action we are considering is 3-dimensional (see Lemma Lemma 4.2.2), it is sensible to look for SU(2)-invariant Cayley submanifolds. Indeed, Theorem 2.3.8 guarantees the local existence and uniqueness of a Cayley passing through any given generic orbit. To construct such a submanifold N, we consider a 1-parameter family of 3-dimensional SU(2)-orbits in M. Hence, the coordinates that do not describe the orbits, i.e. α, β, s, t and δ, need to be functions of a parameter τ. Explicitly, we have:

$$
\begin{aligned}
N =& \Bigg\{ \Bigg((\cos\alpha(\tau)\mathbf{u}, \sin\alpha(\tau)\mathbf{v}), \Bigg((s(\tau)\cos\left(\frac{\delta(\tau)-\gamma}{2}\right), s(\tau)\sin\left(\frac{\delta(\tau)-\gamma}{2}\right), \\
& t(\tau)\cos\left(\frac{\delta(\tau)+\gamma}{2}\right), t(\tau)\sin\left(\frac{\delta(\tau)+\gamma}{2}\right)\Bigg)\Bigg) : |\mathbf{u}| = |\mathbf{v}| = 1, \gamma \in [0, 4\pi), \tau \in (-\epsilon, \epsilon) \Bigg\},
\end{aligned}
\tag{4.2.22}
$$

and its tangent space is spanned by: $\{\partial_1, \partial_2, \partial_3, \dot{s}\partial_s + \dot{t}\partial_t + \dot{\alpha}\partial_\alpha + \dot{\beta}\partial_\beta + \dot{\delta}\partial_\delta\}$, where the dots denotes the derivative with respect to τ. The Cayley condition imposed on this tangent space (see Proposition 2.3.9) generates a system of ODEs in $\alpha, \beta, s, t, \delta$.

Proposition 4.2.12. *Let N be an* SU(2)*-invariant submanifold as described at the beginning of this subsection. Then, N is Cayley in the chart $\mathcal{U}$, defined in Section 4.2.4, if and only if the following system of ODEs is satisfied:*

$$
\begin{cases}
(s^2+t^2)\sin^2\alpha\cos\alpha\dot{\beta} = 0 \\
\cos^2\alpha(t\dot{s} - s\dot{t}) = 0 \\
\cos^2\alpha st\dot{\delta} = 0 \\
5(c+r^2)\cos^2\alpha s\dot{\alpha} - r^2\sin^2\alpha\dot{\alpha}s + 2\sin\alpha\cos\alpha t^2\dot{s} + 4\cos\alpha\sin\alpha s^2\dot{s} + 2\sin\alpha\cos\alpha st\dot{t} = 0 \\
5(c+r^2)\cos^2\alpha t\dot{\alpha} - r^2\sin^2\alpha\dot{\alpha}t + 2\sin\alpha\cos\alpha s^2\dot{t} + 4\cos\alpha\sin\alpha t^2\dot{t} + 2\sin\alpha\cos\alpha st\dot{s} = 0 \\
5(c+r^2)\sin\alpha\cos^2\alpha\dot{\beta}s - 2\sin\alpha\cos\alpha t^2 s\dot{\delta} - r^2\sin^3\alpha\dot{\beta}s = 0 \\
-5(c+r^2)\sin\alpha\cos^2\alpha\dot{\beta}t - 2\sin\alpha\cos\alpha ts^2\dot{\delta} + r^2\sin^3\alpha\dot{\beta}t = 0
\end{cases},
\tag{4.2.23}
$$

where $r^2 = s^2 + t^2$ as usual.

4.2.7.1 Proof of Proposition 4.2.12

In this subsection, we prove Proposition 4.2.12. First, we need to rewrite the tangent space of N in the diagonalizing frame of Section 4.2.6.

Lemma 4.2.13. *The tangent space of N is spanned by:*

$$u := te_{\omega_2} - se_{\omega_1}, \quad v := e_2 + \frac{\sin\alpha}{2}(te_s - se_t), \quad w := e_3 + \sin\alpha\,(te_{\omega_1} + se_{\omega_2})$$

and

$$y := \dot{s}e_s + \dot{t}e_t + \dot{\alpha}e_\alpha + \dot{\beta}e_\beta + \dot{\delta}\,(se_{\omega_1} + te_{\omega_2}).$$

Moreover, through the musical isomorphism, we have:

$$u^\flat = (c+r^2)^{-2/5}(t\omega_2 - s\omega_1), \quad v^\flat = 5(c+r^2)^{3/5}\cos^2\alpha\sigma_2 + 2(c+r^2)^{-2/5}\sin\alpha(t\tilde{ds} - s\tilde{dt}),$$

$$w^\flat = 5(c+r^2)^{3/5}\cos^2\alpha\sigma_3 + (c+r^2)^{-2/5}\sin\alpha\,(t\omega_1 + s\omega_2)$$

and

$$y^\flat = 5(c+r^2)^{3/5}(\dot\alpha d\alpha + \sin^2\alpha\dot\beta d\beta) + 4(c+r^2)^{-2/5}(\dot s\tilde{ds} + \dot t\tilde{dt}) + (c+r^2)^{-2/5}\dot\delta(s\omega_1 + t\omega_2),$$

where $r^2 = s^2 + t^2$.

Proof. One can immediately see from Lemma 4.2.11 that $\partial_1 = u$, $\partial_2 = v$ and $\partial_\delta = se_{\omega_1} + te_{\omega_2}$. We use these equality to obtain:

$$\begin{aligned}(s^2+t^2)\partial_\delta - (t^2-s^2)\partial_1 &= (s^2+t^2)(se_{\omega_1} + te_{\omega_2}) - (t^2-s^2)(te_{\omega_2} - se_{\omega_1})\\ &= 2st(te_{\omega_1} + se_{\omega_2}),\end{aligned}$$

which implies that $\partial_3 = w$. We conclude noticing that $\dot s\partial_s + \dot t\partial_t + \dot\alpha\partial_\alpha + \dot\beta\partial_\beta + \dot\delta\partial_\delta = y - \dot\beta\cos\alpha\partial_1$, where we used once again Lemma 4.2.11. Obviously, the space spanned by $\{u, v, w, y\}$ coincides with the one spanned by $\{u, v, w, y - \dot\beta\cos\alpha\partial_1\}$.

The second part of the Lemma follows immediately from Proposition 4.2.10, where we proved that the metric is diagonal in this frame. □

Let B be as in Proposition 2.3.9. We compute the terms of B in the basis $\{u, v, w, y\}$.

Lemma 4.2.14. *Let u, v, w, y as in Lemma 4.2.13. Then, we have:*

$$\begin{aligned}
B(v,w,y) = {}& 25(c+r^2)^{6/5}\sin\alpha\cos^2\alpha(\dot\beta d\alpha - \dot\alpha d\beta) \\
&+ 2\sin^2\alpha(c+r^2)^{-4/5}\Big((t\dot t + s\dot s)(t\omega_2 - s\omega_1) - (t^2-s^2)\dot\delta(t\tilde{d}t + s\tilde{d}s)\Big) \\
&+ 5(c+r^2)^{1/5}\Bigg(2\cos^2\alpha\big(\dot s\omega_1 - \dot t\omega_2 + \dot\delta(t\tilde{d}t - s\tilde{d}s)\big) \\
&+ 2\cos^2\alpha\sin\alpha(ts\dot\delta\sigma_2 + (s\dot t - t\dot s)\sigma_3) + (s^2+t^2)\sin^3\alpha(\dot\alpha d\beta - \dot\beta d\alpha) \\
&+ 2\sin\alpha\cos\alpha\Big((s^2-t^2)\dot\delta d\alpha + \dot\alpha(t\omega_2 - s\omega_1)\Big) \\
&+ 4\cos\alpha\sin^2\alpha\Big(\dot\beta(s\tilde{d}s + t\tilde{d}t) - (s\dot s + t\dot t)d\beta\Big)\Bigg), \\
B(w,u,y) = {}& 4(c+r^2)^{-4/5}(t^2+s^2)\sin\alpha(\dot t\tilde{d}s - \dot s\tilde{d}t) \\
&+ 5(c+r^2)^{1/5}\big(-2\cos^2\alpha(s\dot s + t\dot t)\sigma_2 - 2\cos\alpha\sin\alpha st\dot\delta d\beta \\
&+ \cos\alpha\sin\alpha\dot\beta(t\omega_1 + s\omega_2) + 2\cos\alpha(s\dot t - t\dot s)d\alpha + 2\cos\alpha\dot\alpha(t\tilde{d}s - s\tilde{d}t) \\
&+ \cos\alpha\sin\alpha(t^2+s^2)\dot\alpha\sigma_2 - \cos\alpha\sin^2\alpha(t^2+s^2)\dot\beta\sigma_3\big), \\
B(u,v,y) = {}& 2(c+r^2)^{-4/5}\sin\alpha(-2\dot\delta st(t\tilde{d}t + s\tilde{d}s) + (t\dot t + s\dot s)(t\omega_1 + s\omega_2)) \\
&+ 5(c+r^2)^{1/5}\big(-2\cos^2\alpha(s\dot s + t\dot t)\sigma_3 - 2\cos\alpha st\dot\delta d\alpha + \cos\alpha\dot\alpha(s\omega_2 + t\omega_1) \\
&+ 2\cos\alpha\sin\alpha(t\dot s - s\dot t)d\beta + 2\cos\alpha\sin\alpha\dot\beta(s\tilde{d}t - t\tilde{d}s) \\
&+ (s^2+t^2)\cos\alpha\sin\alpha\dot\alpha\sigma_3 + (s^2+t^2)\cos\alpha\sin^2\alpha\dot\beta\sigma_2\big), \\
B(v,u,w) = {}& 2(c+r^2)^{-4/5}\sin^2\alpha(t^2+s^2)(t\tilde{d}t + s\tilde{d}s) \\
&+ 10(c+r^2)^{1/5}\big(-\cos^2\alpha(s\tilde{d}s + t\tilde{d}t) + \sin\alpha\cos\alpha(t^2+s^2)d\alpha\big),
\end{aligned}$$

where B is defined in Proposition 2.3.9 and $r^2 = s^2 + t^2$.

Proof. The multilinearity of the Cayley form Φ_c implies that the same property holds for B. Now, expanding the formula (Eq. (4.2.19)) for Φ_c, we obtain:

$$\begin{aligned}
\Phi_c = {}& 4(c+r^2)^{-4/5}\tilde{d}s\wedge\tilde{d}t\wedge\omega_2\wedge\omega_1 + 25(c+r^2)^{6/5}\sin\alpha\cos^2\alpha d\alpha\wedge d\beta\wedge\sigma_3\wedge\sigma_2 \\
& 10(c+r^2)^{1/5}\Bigg(\sin\alpha\tilde{d}s\wedge\omega_1\wedge d\alpha\wedge d\beta + \cos^2\alpha\tilde{d}s\wedge\omega_1\wedge\sigma_2\wedge\sigma_3 \\
& - \sin\alpha\tilde{d}t\wedge\omega_2\wedge d\alpha\wedge d\beta - \cos^2\alpha\tilde{d}t\wedge\omega_2\wedge\sigma_2\wedge\sigma_3 + 2\cos\alpha\tilde{d}s\wedge\tilde{d}t\wedge d\alpha\wedge\sigma_2 \\
& + \frac{\cos\alpha}{2}\omega_1\wedge\omega_2\wedge d\alpha\wedge\sigma_2 - \frac{\cos\alpha\sin\alpha}{2}\omega_1\wedge\omega_2\wedge d\beta\wedge\sigma_3 - \cos\alpha\tilde{d}s\wedge\omega_2\wedge d\alpha\wedge\sigma_3 \\
& - 2\cos\alpha\sin\alpha\tilde{d}s\wedge\tilde{d}t\wedge d\beta\wedge\sigma_3 - \cos\alpha\sin\alpha\tilde{d}s\wedge\omega_2\wedge d\beta\wedge\sigma_2 \\
& - \cos\alpha\tilde{d}t\wedge\omega_1\wedge d\alpha\wedge\sigma_3 - \cos\alpha\sin\alpha\tilde{d}t\wedge\omega_1\wedge d\beta\wedge\sigma_2\Bigg).
\end{aligned}$$

It is straightforward to conclude using the definition of B. ☐

Consider the two-form given in Proposition 2.3.9 that projects to η through π_7. The summands of such two form can be computed through a direct computation involving the terms obtained in Lemma 4.2.13 and Lemma 4.2.14.

Corollary 4.2.15. *Let u, v, w, y as in Lemma 4.2.13 and let $\Psi_1 := u^\flat \wedge B(v, w, y)$, $\Psi_2 = v^\flat \wedge B(w, u, y)$, $\Psi_3 = w^\flat \wedge B(u, v, y)$, $\Psi_4 = y^\flat \wedge B(v, u, w)$, where B is as defined in Proposition 2.3.9. Then, we have:*

$$\begin{aligned}
\Psi_1 = &25(c+r^2)^{4/5}\sin\alpha\cos^2\alpha(t\omega_2 - s\omega_1)\wedge(\dot\beta d\alpha - \dot\alpha d\beta)\\
&-(c+r^2)^{-6/5}2\sin^2\alpha(t^2-s^2)\dot\delta(t\omega_2 - s\omega_1)\wedge(t\tilde{d}t + s\tilde{d}s)\\
&+5(c+r^2)^{-1/5}\Big(2\cos^2\alpha\Big((t\dot s - s\dot t)\omega_2\wedge\omega_1 + \dot\delta(t\omega_2 - s\omega_1)\wedge(t\tilde{d}t - s\tilde{d}s)\Big)\\
&+2\sin\alpha\cos^2\alpha\Big(ts\dot\delta(t\omega_2 - s\omega_1)\wedge\sigma_2 + (s\dot t - t\dot s)(t\omega_2 - s\omega_1)\wedge\sigma_3\Big)\\
&+2\sin\alpha\cos\alpha(s^2-t^2)\dot\delta(t\omega_2 - s\omega_1)\wedge d\alpha\\
&+(t^2+s^2)\sin^3\alpha(t\omega_2 - s\omega_1)\wedge(\dot\alpha d\beta - \dot\beta d\alpha)\\
&+4\cos\alpha\sin^2\alpha\Big(\dot\beta(t\omega_2 - s\omega_1)\wedge(s\tilde{d}s + t\tilde{d}t) - (s\dot s + t\dot t)(t\omega_2 - s\omega_1)\wedge d\beta\Big)\Big),\\
\Psi_2 = &25(c+r^2)^{4/5}\Big(-2\cos^3\alpha\sin\alpha st\dot\delta\sigma_2\wedge d\beta + \cos^3\alpha\sin\alpha\dot\beta\sigma_2\wedge(t\omega_1 + s\omega_2)\\
&+2\cos^3\alpha(s\dot t - t\dot s)\sigma_2\wedge d\alpha + 2\cos^3\alpha\dot\alpha\sigma_2\wedge(t\tilde{d}s - s\tilde{d}t)\\
&-\cos^3\alpha\sin^2\alpha(t^2+s^2)\dot\beta\sigma_2\wedge\sigma_3\Big)\\
&+10(c+r^2)^{-1/5}\Big(2\sin\alpha\cos^2\alpha(t^2+s^2)\sigma_2\wedge(\dot t\tilde{d}s - \dot s\tilde{d}t)\\
&-2\cos\alpha\sin^2\alpha st\dot\delta(t\tilde{d}s - s\tilde{d}t)\wedge d\beta + \cos\alpha\sin^2\alpha\dot\beta(t\tilde{d}s - s\tilde{d}t)\wedge(t\omega_1 + s\omega_2)\\
&+2\cos\alpha\sin\alpha(s\dot t - t\dot s)(t\tilde{d}s - s\tilde{d}t)\wedge d\alpha - \cos\alpha\sin^2\alpha(t^s + s^2)\dot\alpha\sigma_2\wedge(t\tilde{d}s - s\tilde{d}t)\\
&+\cos\alpha\sin^3\alpha(t^2+s^2)\dot\beta\sigma_3\wedge(t\tilde{d}s - s\tilde{d}t)\Big)\\
&+8(c+r^2)^{-6/5}\sin^2\alpha(t^2+s^2)(s\dot t - t\dot s)\tilde{d}s\wedge\tilde{d}t,
\end{aligned}$$

$$
\begin{aligned}
\Psi_3 = {} & 25(c+r^2)^{4/5}\Big(-2\cos^3\alpha st\dot{\delta}\sigma_3\wedge d\alpha + \cos^3\alpha\dot{\alpha}\sigma_3\wedge(s\omega_2+t\omega_1) \\
& + 2\cos^3\alpha\sin\alpha(t\dot{s}-s\dot{t})\sigma_3\wedge d\beta + 2\cos^3\alpha\sin\alpha\dot{\beta}\sigma_3\wedge(s\tilde{dt}-t\tilde{ds}) \\
& + (s^2+t^2)\cos^3\alpha\sin^2\alpha\dot{\beta}\sigma_3\wedge\sigma_2\Big) \\
& - 4(c+r^2)^{-6/5}\sin^2\alpha\dot{\delta}st(t\omega_1+s\omega_2)\wedge(t\tilde{dt}+s\tilde{ds}) \\
& + 5(c+r^2)^{-1/5}\Big(2\sin\alpha\cos^2\alpha\left((t\dot{t}+s\dot{s})\sigma_3\wedge(t\omega_1+s\omega_2) - 2\dot{\delta}st\sigma_3\wedge(t\tilde{dt}+s\tilde{ds})\right) \\
& - 2\sin\alpha\cos^2\alpha(s\dot{s}+t\dot{t})(t\omega_1+s\omega_2)\wedge\sigma_3 - 2\cos\alpha\sin\alpha st\dot{\delta}(t\omega_1+s\omega_2)\wedge d\alpha \\
& + 2\cos\alpha\sin^2\alpha(t\dot{s}-s\dot{t})(t\omega_1+s\omega_2)\wedge d\beta + 2\cos\alpha\sin^2\alpha\dot{\beta}(t\omega_1+s\omega_2)\wedge(s\tilde{dt}-t\tilde{ds}) \\
& + (s^2+t^2)\cos\alpha\sin^2\alpha\dot{\alpha}(t\omega_1+s\omega_2)\wedge\sigma_3 + (s^2+t^2)\cos\alpha\sin^3\alpha\dot{\beta}(t\omega_1+s\omega_2)\wedge\sigma_2\Big), \\
\Psi_4 = {} & 2(c+r^2)^{-6/5}\sin^2\alpha(t^2+s^2)\left(\dot{\delta}(s\omega_1+t\omega_2)\wedge(t\tilde{dt}+s\tilde{ds}) + 4(\dot{t}s-\dot{s}t)\tilde{dt}\wedge\tilde{ds}\right) \\
& + 50(c+r^2)^{4/5}\Big(-\cos^2\alpha(\dot{\alpha}d\alpha+\sin^2\alpha\dot{\beta}d\beta)\wedge(s\tilde{ds}+t\tilde{dt}) \\
& + \cos\alpha\sin^3\alpha(t^2+s^2)\dot{\beta}d\beta\wedge d\alpha\Big) \\
& + 10(c+r^2)^{-1/5}\Big(\sin^2\alpha(t^2+s^2)(\dot{\alpha}d\alpha+\sin^2\alpha\dot{\beta}d\beta)\wedge(t\tilde{dt}+s\tilde{ds}) \\
& - 4\cos^2\alpha(\dot{s}t-\dot{t}s)\tilde{ds}\wedge\tilde{dt} \\
& + 4\sin\alpha\cos\alpha(t^2+s^2)(\dot{s}\tilde{ds}+\dot{t}\tilde{dt})\wedge d\alpha - \cos^2\alpha\dot{\delta}(s\omega_1+t\omega_2)\wedge(s\tilde{ds}+t\tilde{dt}) \\
& + \sin\alpha\cos\alpha(t^2+s^2)\dot{\delta}(s\omega_1+t\omega_2)\wedge d\alpha\Big),
\end{aligned}
$$

where $r^2 = s^2+t^2$.

Moreover,

$$\eta = \pi_7(\Psi_1+\Psi_2+\Psi_3+\Psi_4),$$

where η *and* π_7 *are defined in Proposition 2.3.9.*

Finally, we turn our attention to the map π_7. As recalled in Remark 2.3.10, this map is the projection to the linear subspace Λ^2_7 of the space of 2-forms on M.

Lemma 4.2.16. *In the coframe* $\{\sigma_2, \sigma_3, d\alpha, d\beta, \omega_1, \omega_2, \tilde{ds}, \tilde{dt}\}$*, a basis for* Λ^2_7 *is given by* *the following 2-forms:*

$$\begin{aligned}
\lambda_1 &:= -\cos\alpha\sigma_2\wedge\omega_1 + d\alpha\wedge\omega_2 + 2\sin\alpha d\beta\wedge\tilde{dt} + 2\cos\alpha\sigma_3\wedge\tilde{ds},\\
\lambda_2 &:= \cos\alpha\sigma_2\wedge\omega_2 + d\alpha\wedge\omega_1 - 2\sin\alpha d\beta\wedge\tilde{ds} + 2\cos\alpha\sigma_3\wedge\tilde{dt},\\
\lambda_3 &:= \cos\alpha\sigma_3\wedge\omega_1 + \sin\alpha d\beta\wedge\omega_2 + 2\cos\alpha\sigma_2\wedge\tilde{ds} - 2d\alpha\wedge\tilde{dt},\\
\lambda_4 &:= -\cos\alpha\sigma_3\wedge\omega_2 + \sin\alpha d\beta\wedge\omega_1 + 2\cos\alpha\sigma_2\wedge\tilde{dt} + 2d\alpha\wedge\tilde{ds},\\
\lambda_5 &:= 5(c+r^2)\cos\alpha\sigma_3\wedge d\alpha + 5(c+r^2)\sin\alpha\cos\alpha\sigma_2\wedge d\beta + 2\omega_2\wedge\tilde{ds} + 2\omega_1\wedge\tilde{dt},\\
\lambda_6 &:= 5(c+r^2)\sin\alpha\cos\alpha\sigma_3\wedge d\beta - 5(c+r^2)\cos\alpha\sigma_2\wedge d\alpha + \omega_2\wedge\omega_1 + 4\tilde{dt}\wedge\tilde{ds},\\
\lambda_7 &:= 5(c+r^2)\sin\alpha d\beta\wedge d\alpha + 5(c+r^2)\cos^2\alpha\sigma_3\wedge\sigma_2 + 2\tilde{ds}\wedge\omega_1 - 2\tilde{dt}\wedge\omega_2.
\end{aligned}$$

Proof. Using the explicit formula for π_7 given in Proposition 2.3.9, it is easy to verify that $\pi_7(\lambda_i) = \lambda_i$ for all $i = 1...7$. We deduce that the λ_is form a basis of Λ^7_2 as they are linearly independent and the dimension of Λ^7_2 is 7. □

At this point, the proof of Proposition 4.2.12 follows easily. Indeed, we can rewrite the sum of the Ψ_i given in Corollary 4.2.15 as follows:

$$\begin{aligned}
\Psi_1 + \Psi_2 + &\Psi_3 + \Psi_4 =\\
=&5(c+r^2)^{-1/5}\left(-5(c+r^2)\sin\alpha\cos^2\alpha\dot{\beta}t + r^2\sin^3\alpha\dot{\beta}t - 2\sin\alpha\cos\alpha ts^2\dot{\delta}\right)\lambda_1\\
&+ 5(c+r^2)^{-1/5}\Big(5(c+r^2)\sin\alpha\cos^2\alpha\dot{\beta}s - r^2\sin^3\alpha\dot{\beta}s - 2\sin\alpha\cos\alpha t^2 s\dot{\delta}\Big)\lambda_2\\
&+ 5(c+r^2)^{-1/5}\Bigg(5(c+r^2)\cos^2\alpha t\dot{\alpha} + 4\cos\alpha\sin\alpha t^2\dot{t} + 2\sin\alpha\cos\alpha st\dot{s}\\
&+ 2\sin\alpha\cos\alpha s^2\dot{t} - r^2\sin^2\alpha\dot{\alpha}t\Bigg)\lambda_3 + 5(c+r^2)^{-1/5}\Bigg(-5(c+r^2)\cos^2\alpha s\dot{\alpha}\\
&- 4\cos\alpha\sin\alpha s^2\dot{s} - 2\sin\alpha\cos\alpha st\dot{t} - 2\sin\alpha\cos\alpha t^2\dot{s} + r^2\sin^2\alpha\dot{\alpha}s\Bigg)\lambda_4\\
&- 2\cos^2\alpha st\dot{\delta}\left(25(c+r^2)^{-1/5}\lambda_5\right)\\
&+ 2\cos^2\alpha(t\dot{s} - s\dot{t})\left(25(c+r^2)^{-1/5}\lambda_6\right)\\
&+ 2(s^2+t^2)\sin^2\alpha\cos\alpha\dot{\beta}\left(25(c+r^2)^{-1/5}\lambda_7\right).
\end{aligned}$$

From Corollary 4.2.15 and Lemma 4.2.16, we deduce the ODEs of Proposition 4.2.12.

Corollary 4.2.17. *Let N be an $\mathrm{SU}(2)$-invariant submanifold as described at the beginning of this subsection. Then, N is Cayley in the chart $\mathcal{U}$, defined in Section 4.2.4, if and only* *if the following system of ODEs is satisfied:*

$$\begin{cases} \dot{\beta} = 0 \\ (t\dot{s} - s\dot{t}) = 0 \\ \dot{\delta} = 0 \\ 5(c + r^2)\cos^2\alpha st\dot{\alpha} - (s^2 + t^2)st\sin^2\alpha\dot{\alpha} + 2\sin\alpha\cos\alpha(s^2 + t^2)(s\dot{t} + t\dot{s}) = 0 \end{cases},$$

where $r^2 = s^2 + t^2$ as usual.

Proof. As $\alpha \in (0, \pi/2)$ and $s, t > 0$, we get immediately the first three equations from the first three equations of Eq. (4.2.23). The last two equations of Eq. (4.2.23) are superfluous as $\dot{\beta} = 0$ and $\dot{\delta} = 0$. The same holds for t times the fourth equation plus s times the fifth equation of Eq. (4.2.23), where we use $t\dot{s} - s\dot{t} = 0$ this time. We conclude by considering s times the fifth equation minus t times the fourth equation of Eq. (4.2.23). □

4.2.8 Cayley fibration

In the previous section we found the condition that makes N, SU(2)-invariant submanifold, a Cayley submanifold. Explicitly, it consists of a system of ODEs that is completely integrable; these solutions will give us the desired fibration.

Proposition 4.2.18. *Let N be an* SU(2)*-invariant submanifold as described at the beginning of Section 4.2.7. Then, N is Cayley in $\mathcal{U}$, defined in Section 4.2.4, if and only if the following quantities are constant:*

$$\beta, \qquad \delta, \qquad \frac{s}{t}, \qquad F := 2\sin^{5/2}\alpha\cos^{1/2}\alpha st + 5c\frac{st}{(s^2 + t^2)}H(\alpha),$$

where $H(\alpha)$ is the primitive function of $h(\alpha) := (\cos\alpha\sin\alpha)^{3/2}$.

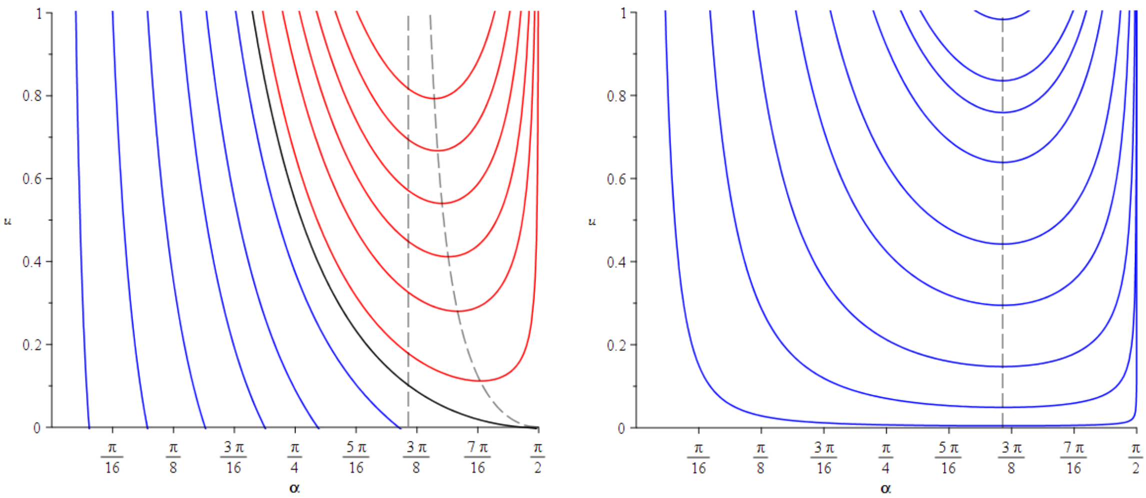
Proof. The condition on β and δ follows immediately from Corollary 4.2.17. Taking the derivative in τ of s/t, we see that

$$0 = \frac{d}{d\tau}\left(\frac{s}{t}\right) = \frac{\dot{s}t - \dot{t}s}{t^2},$$

which is equivalent to the second equation in Corollary 4.2.17, as $t > 0$. Analogously, one can see that the derivative with respect to τ of F is equivalent to the last equation of Corollary 4.2.17 if we assume that s/t is constant. □

Setting

$$v := \frac{s}{t}, \qquad u := st,$$



(a) Level sets of F with $c = 1$ and $v = 1$ (b) Level sets of F with $c = 0$ and $v = 1$

Figure 4.1: Level sets of F in the generic and in the conical case

the preserved quantities transform to:

$$\beta, \qquad \delta, \qquad v, \qquad F := 2\sin^{5/2}\alpha\cos^{1/2}\alpha(v^2+1)u + 5cvH(\alpha),$$

where we multiplied F by the constant (v^2+1). Observe that this is an admissible transformation from $s, t \in (0, \infty)$ to $u, v \in (0, \infty)$. Moreover, fixed β, δ, v, we can represent the SU(2)-invariant Cayley submanifolds as the level sets of F reckoned as a $\mathbb{R}$-valued function of α and u. An easy analysis of F shows that these level sets can be represented as in Fig. 4.1. The dashed lines in the two graphs correspond to the curves formed by the u-minimums of each level set and to the two vertical lines: $\alpha = \arccos(1/\sqrt{6})$. For $c = 0$, these coincide, while in the generic case the locus of the u-minimum is:

$$\alpha = \arccos\left(\sqrt{\frac{u(v^2+1)}{6u(v^2+1)+5cv}}\right),$$

which is only asymptotic to $\alpha = \arccos(1/\sqrt{6})$ for $u \to \infty$.

The conical version. We first consider the easier case, i.e. when $c = 0$. It is clear from the graph that the SU(2)-invariant Cayleys passing through $\mathcal{U}$ are contained in $\mathcal{U}$, have topology $S^3 \times \mathbb{R}$ and are smooth. Moreover, we can construct a Cayley fibration on the chart $\mathcal{U}$ with base an open subset of $\mathbb{R}^4$. To do so, we associate to each point of $\mathcal{U}$ the value of β, δ, s/t and F of the Cayley passing through that point. This SU(2)-invariant fibration naturally extends to the whole M_0 via continuity. Using Table 4.1 and Theorem 2.3.8, we can describe the extension precisely. Indeed, when $\alpha = \pi/2$, the fibres of π_{S^4} are SU(2)-invariant Cayley submanifolds; when $\alpha \neq \pi/2$ and $s = 0$ or

$t = 0$, the suitable Cayley submanifolds constructed by Karigiannis and Min-Oo [50] are SU(2)-invariant; finally, when $\alpha = 0$ and $(s,t) \neq 0$, the fibres are given by an extension of [48]. We recall that the Karigiannis–Min-Oo Cayley submanifolds are constructed as vector subbundles of $\slashed{S}_-(S^4)$ over a minimal surface of S^4. The topology of these Cayley submanifolds that are not contained in $\mathcal{U}$ is $\mathbb{R}^4 \setminus \{0\}$ in the first case and $\mathbb{R} \times S^3$ in the remaining ones. Observe that this fibration does not admit singular or intersecting fibres.

The smooth version. Now, we consider the generic case, i.e. when $c > 0$. Differently from the cone, the graph of the level sets of F shows that the SU(2)-invariant Cayley submanifolds passing through $\mathcal{U}$ do not remain contained in it, and they admit three different topologies in the extension. The red, black and blue lines correspond to submanifolds with topology $\mathbb{R} \times S^3$, $\mathbb{R}^4$ and $\mathcal{O}_{\mathbb{CP}^1}(-1)$, respectively. We define an SU(2)-invariant Cayley fibration on $\mathcal{U}$ that extends to the whole M exactly as above. If we fix a value of F corresponding to a Cayley of topology $\mathcal{O}_{\mathbb{CP}^1}(-1)$, then, for every δ, v, all the different Cayleys will intersect in a $\mathbb{CP}^1 \subset S^4$, where S^4 is the zero section of $\slashed{S}_-(S^4)$.

The parametrizing space. Using Fig. 4.1, we can study the parametrizing space $\mathcal{B}$ of the Cayley fibrations we have just described. We will only deal with the smooth version, as the conical case is going to be completely analogous.

Ignoring β for a moment, it is immediate to see that, if we restrict our attention to the fibres that are topologically $\mathcal{O}_{\mathbb{CP}^1}(-1)$ and the ones corresponding to the black line, the parametrizing space is homeomorphic to $S^2 \times [0,1]$. The remaining fibres are parametrized by $B^3(1)$, open unit ball of $\mathbb{R}^3$. As we removed the zero section of $\slashed{S}_-(S^4)$, it is clear that we can glue these partial parametrizations together to obtain $\overline{B^3(2)}$. Now, β gives a circle action on $\overline{B^3(2)}$ that vanishes on its boundary. We conclude that the parametrizing space $\mathcal{B}$ of the smooth Cayley fibration is S^4. Indeed, this is essentially the same way to describe S^4 as we did in Section 4.2.1.

The smoothness of the fibres (the asymptotic analysis as $r \to 0$). In this subsection, we study the smoothness of the fibres. Observe that this property is obviously satisfied as long as they are contained in the chart $\mathcal{U}$. Hence, the Cayleys of topology $S^3 \times \mathbb{R}$ are smooth, and we only need to check the remaining ones in the points where they meet the zero section, i.e., when the SU(2) group action degenerates. To this purpose, we carry out an asymptotic analysis.

Let β_0, v_0, δ_0 and F_0 be the constants determining a Cayley fibre N. By the explicit formula for F, we see that N is given by:

$$u = \frac{F_0 - 5cv_0 H(\alpha)}{2\sin^{5/2}\alpha \cos^{1/2}\alpha (v_0^2 + 1)}$$

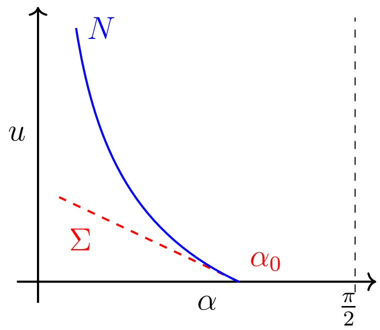


Figure 4.2: Approximation of a Cayley at $u = 0$ when $\alpha_0 \in (0, \pi/2)$

We first check the smoothness of the fibres that meet the zero section ($u = 0$) at some $\alpha_0 \in (0, \pi/2)$, i.e., the ones of topology $\mathcal{O}_{\mathbb{CP}^1}(-1)$. For this purpose, if we expand near α_0^- and we obtain the linear approximation of N at that point. Explicitly, this is the SU(2)-invariant 4-dimensional submanifold Σ characterized by the equation

$$u = -\frac{5cv_0}{2\tan\alpha_0(v_0^2+1)}(\alpha - \alpha_0),$$

and where v, δ, β are constantly equal to v_0, δ_0, β_0.

Now, we want to study the asymptotic behaviour of the metric g_c when restricted to Σ, and then, we let α tends to α_0 from the left. To do so, it is convenient to compute the following identities using the definition of $u := st$ and $v := s/t$:

$$\begin{aligned} dt &= \frac{1}{2\sqrt{uv}}du - \frac{1}{2v}\sqrt{\frac{u}{v}}dv, \\ ds &= \frac{\sqrt{v}}{2\sqrt{u}}du + \frac{\sqrt{u}}{2\sqrt{v}}dv, \\ ds^2 &= \frac{v}{4u}du^2 + \frac{u}{4v}dv^2 + \frac{1}{2}dudv, \\ dt^2 &= \frac{1}{4uv}du^2 + \frac{u}{4v^3}dv^2 - \frac{1}{2v^2}dudv. \end{aligned} \tag{4.2.24}$$

The metric g_c, in the coframe $\{\sigma_1, \sigma_2, \sigma_3, d\alpha, d\beta, du, dv, d\delta\}$, then can be rewritten as:

$$\begin{aligned}
g_c =& 5\left(c+\frac{u}{v}(1+v^2)\right)^{3/5}\left(d\alpha^2+\sin^2\alpha d\beta^2+\cos^2\alpha(\sigma_2^2+\sigma_3^2)\right)\\
&+4\left(c+\frac{u}{v}(1+v^2)\right)^{-2/5}\Bigg(\frac{1}{4uv}(1+v^2)du^2+\frac{u}{4v^3}(1+v^2)dv^2+\frac{1}{2v^2}(v^2-1)dudv\\
&+\frac{u}{v}(1+v^2)\frac{\cos^2\alpha}{4}d\beta^2+\frac{u}{4v}(1+v^2)\sigma_1^2-\frac{\cos\alpha}{2}\frac{u}{v}(1+v^2)d\beta\sigma_1\\
&+\frac{u}{v}(1+v^2)\frac{\sin^2\alpha}{4}(\sigma_2^2+\sigma_3^2)+\frac{u(1-v^2)}{2v}d\delta\sigma_1+u\sin\alpha d\delta\sigma_3+\frac{u}{4v}(1+v^2)d\delta^2\\
&+\sin\alpha\frac{u}{v}dv\sigma_2-\frac{u(1-v^2)\cos\alpha}{2v}d\delta d\beta\Bigg),
\end{aligned} \tag{4.2.25}$$

where we used Eq. (4.2.24) and Lemma 4.2.7. Now, if we restrict Eq. (4.2.25) to Σ, and we let α tend to α_0 from the left, we get:

$$\begin{aligned}
g_c\big|_N &\sim \frac{c^{-2/5}}{v_0}(1+v_0^2)\left(\frac{du^2}{u}+u\sigma_1^2\right)+5c^{3/5}\cos^2\alpha_0(\sigma_2^2+\sigma_3^2)\\
&\sim dr^2+r^2\frac{\sigma_1^2}{4}+5c^{3/5}\cos^2\alpha_0(\sigma_2^2+\sigma_3^2),
\end{aligned}$$

where

$$r=\sqrt{\frac{1+v_0^2}{v_0c^{2/5}}}2\sqrt{u}.$$

As the length of σ_1 is 4π, we deduce that the metric g_c extends smoothly to the $\mathbb{CP}^1 \cong S^2$ contained in the zero section. This two-dimensional sphere corresponds to the base of the bundle $\mathcal{O}_{\mathbb{CP}^1}(-1)$.

Finally, we check the smoothness of the fibres meeting the zero section at $\alpha_0 = \pi/2$, i.e., the ones with topology $\mathbb{R}^4$. Expanding for $\alpha \to \pi/2^-$, we immediately see that the first order is not enough and we need to pass to second order. Explicitly, this is the SU(2)-invariant 4-dimensional submanifold Σ of equation:

$$u = A(\alpha-\pi/2)^2,$$

where $A := cv(1+v^2)^{-1}$ is the constant depending on c, v determined by the expansion. As above, the remaining parameters v, δ, β are constantly equal to v_0, δ_0, β_0. If we restrict g_c as defined in Eq. (4.2.25) to Σ, and we let α tend to $\pi/2$, then, we obtain:

$$\begin{aligned}
g_c\big|_N \sim\ & 5c^{3/5}(\alpha-\pi/2)^2(\sigma_2^2+\sigma_3^2)+Ac^{-2/5}\left(\frac{1+v^2}{v}\right)(\alpha-\pi/2)^2(\sigma_1^2+\sigma_2^2+\sigma_3^2)\\
&+\left(5c^{3/5}+4Ac^{-2/5}\frac{1+v^2}{v}\right)d\alpha^2\\
\sim\ & c^{3/5}(\alpha-\alpha_0)^2\left(\sigma_1^2+6(\sigma_2^2+\sigma_3^2)\right)+9c^{3/5}d\alpha^2,
\end{aligned}$$

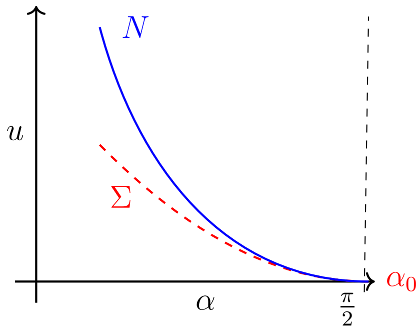


Figure 4.3: Approximation of a Cayley at $u = 0$ when $\alpha_0 = \pi/2$

where we also used the expansion of $\cos\alpha$ around $\pi/2$ and the explicit value of A. We conclude that N is not smooth when it meets the zero section, and it develops an asymptotically conical singularity at that point.

Remark 4.2.19. The singularity is asymptotic to the Lawson–Osserman cone [55].

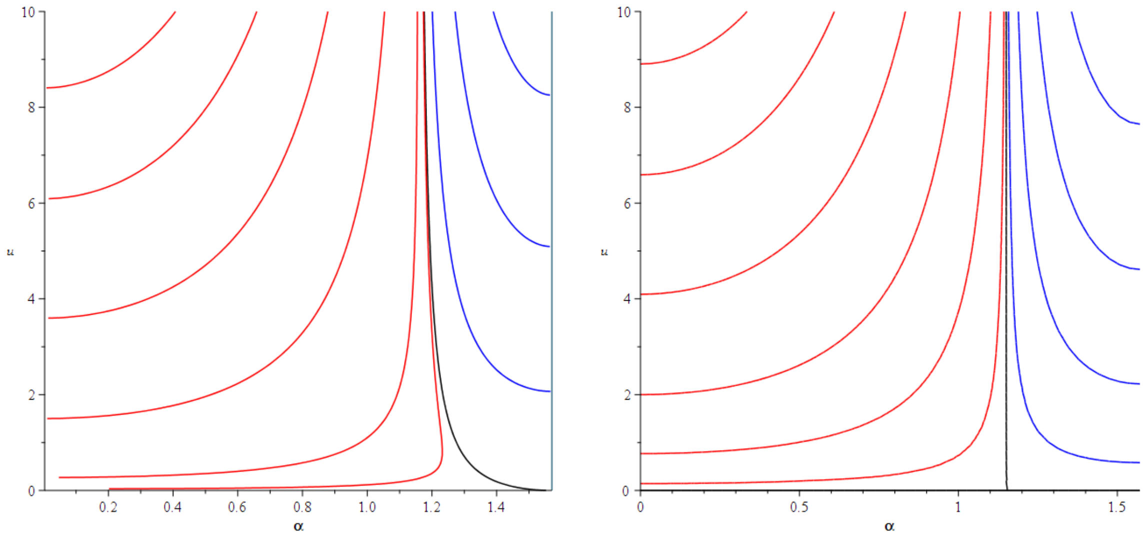
The main theorems We collect all these results in the following theorems. Observe that we are using the notion of Cayley fibration given in Definition 3.1.5.

Theorem 4.2.20 (T. [71]; Generic case)**.** *Let (M, Φ_c) be the Bryant–Salamon manifold constructed over the round sphere S^4 for some $c > 0$, and let* $\mathrm{SU}(2)$ *act on M as in Section 4.2.3. Then, M admits an* $\mathrm{SU}(2)$*-invariant Cayley fibration parametrized by $\mathcal{B} \cong S^4$. The fibres are topologically $\mathcal{O}_{\mathbb{CP}^1}(-1)$, $S^3 \times \mathbb{R}$ and $\mathbb{R}^4$. Apart from the non-vertical fibres of topology $\mathbb{R}^4$, all the others are smooth. The singular fibres of the Cayley fibration have a conically singular point and are parametrized by $(\mathcal{B}^\circ)^c \cong S^2 \times S^1$ (β, δ, v in our description). Moreover, at each point of the zero section $S^4 \subset \slashed{S}_-(S^4)$, infinitely many Cayley fibres intersect.*

Theorem 4.2.21 (T. [71]; Conical case)**.** *Let (M_0, Φ_0) be the conical Bryant–Salamon manifold constructed over the round sphere S^4, and let* $\mathrm{SU}(2)$ *act on M_0 as in Section 4.2.3. Then, M_0 admits an* $\mathrm{SU}(2)$*-invariant Cayley fibration parametrized by $\mathcal{B} \cong S^4$. The fibres are topologically $S^3 \times \mathbb{R}$ and are all smooth. Moreover, as these do not intersect, the* $\mathrm{SU}(2)$*-invariant Cayley fibration is a fibration in the usual differential geometric sense with fibres Cayley submanifolds.*

Remark 4.2.22. It is interesting to observe that, in the generic case, the family of singular $\mathbb{R}^4$s separates the fibres of topology $S^3 \times \mathbb{R}$ from the ones of topology $\mathcal{O}_{\mathbb{CP}^1}(-1)$.

Remark 4.2.23. Similarly to [49, Subsection 5.11.1], one can blow-up at the north pole and argue that in the limit the Cayley fibration splits into the product of a line $\mathbb{R}$ and



(a) Level sets of ν_1 with $v = 1$ (b) Level sets of ν_0 with $v = 1$

Figure 4.4: Level sets of the multi-moment map in the generic and conical case

of an SU(2)-invariant coassociative fibration on $\mathbb{R}^7$. By the uniqueness of the SU(2)-invariant coassociative fibrations of $\mathbb{R}^7$, we deduce that the latter is the Harvey and Lawson coassociative fibration [37, Section IV.3] up to a reparametrization.

Remark 4.2.24. From the computations that we have carried out, it is easy to give an explicit formula for the multi-moment map ν_c associated to this action. Indeed, this is:

$$\nu_c = 5(c + s^2 + t^2)^{1/5} \left((s^2 + t^2) \cos^2 \alpha - \frac{1}{6}(s^2 + t^2 - 5c) \right) - \frac{25}{6} c^{6/5} \qquad c \geq 0.$$

Obviously, the range of ν_c is the whole $\mathbb{R}$. Under the usual transformation $u = st$ and $v = s/t$, the multi-moment map becomes:

$$\nu_c = \frac{5}{6} \left(c + \frac{u(1+v^2)}{v} \right)^{1/5} \left(6 \frac{u(1+v^2)}{v} \cos^2 \alpha - \frac{u(1+v^2)}{v} + 5c \right) - \frac{25}{6} c^{6/5}.$$

We draw the level sets of ν_c in Figure Fig. 4.4.

The black lines correspond to the level set relative to zero, the red lines correspond to negative values, while the blue lines correspond to the positive ones.

Differently from the conical case, the 0-level set of ν_c for $c > 0$ does not coincide with the locus of u-minimum of each level set of F. Moreover, for every $c \geq 0$, it does not even coincide with the set of SU(2)-orbits of minimum volume in each fibre.

Asymptotic geometry as $r \to \infty$. Inspecting the geometry of the Cayley fibration (see Fig. 4.1), we deduce that there are two asymptotic behaviours for the fibres: one for $\alpha \sim 0$ and one for $\alpha \sim \pi/2$. In both cases, as $u \to \infty$, the tangent space of the Cayley fibre N tends to be spanned by $\partial_u, \partial_1, \partial_2, \partial_3$. We can use the formula for the metric (Eq. (4.2.25)) to obtain, for $\alpha \sim 0$:

$$
\begin{aligned}
g_c\big|_N &\sim 5\left(\frac{1+v^2}{v}\right)^{3/5} u^{3/5}(\sigma_2^2+\sigma_3^2) + u^{-2/5}\left(\frac{1+v^2}{v}\right)^{-2/5}\left(\frac{1+v^2}{v}\right)\left(\frac{du^2}{u}+u\sigma_1^2\right) \\
&= \left(\frac{1+v^2}{v}\right)^{3/5}\left(u^{3/5}(5(\sigma_2^2+\sigma_3^2)+\sigma_1^2)+u^{-7/5}du^2\right) \\
&= dr^2 + \frac{9}{25}r^2\frac{(\sigma_1^2+5(\sigma_2^2+\sigma_3^2))}{4},
\end{aligned}
$$

and, for $\alpha \sim \pi/2$:

$$
\begin{aligned}
g_c\big|_N &\sim \left(\frac{1+v^2}{v}\right)^{3/5}\left(u^{3/5}(\sigma_1^2+\sigma_2^2+\sigma_3^2)+u^{-7/5}du^2\right) \\
&= dr^2 + \frac{9}{25}r^2\frac{(\sigma_1^2+\sigma_2^2+\sigma_3^2)}{4},
\end{aligned}
$$

where, in both cases,

$$
r := \frac{10}{3}\left(\frac{1+v^2}{v}\right)^{3/10} u^{3/10}.
$$

When $\alpha \sim \pi/2$, the link S^3 is endowed with the round metric, while, when $\alpha \sim 0$, the round sphere is squashed by a factor $1/5$.

Remark 4.2.25. Observe that $1/5$ is also the squashing factor on the round metric of S^7 that makes the space homogeneous, non-round and Einstein. It is well-known that there are no other metrics satisfying these properties [73].

4.3 Cayley fibration invariant under the lift of the $\mathrm{Sp}(1)\times \mathrm{Id}_1$-action on S^4

Let $M := \$_-(S^4)$ and $M_0 := \mathbb{R}^+ \times S^7$ be endowed with the torsion-free Spin(7)-structures Φ_c constructed by Bryant and Salamon that we described in Section 2.3.3. On each Spin(7) manifold, we construct the Cayley Fibration which is invariant under the lift to M (or M_0) of the standard (left multiplication) $\mathrm{Sp}(1) \times \mathrm{Id}_1$-action on $S^4 \subset \mathbb{H} \oplus \mathbb{R}$.

Remark 4.3.1. The exact same computations will work for the $\mathrm{Sp}(1) \times \mathrm{Id}_1$-action given by right multiplication of the quaternionic conjugate. In this case, the role of the north and of the south pole will be interchanged.

4.3.1 Choice of coframe on S^4

As in Section 4.2, we choose an adapted orthonormal coframe on S^4 which is compatible with the symmetries we will impose.

Consider $\mathbb{R}^5$ as the sum of a 4-dimensional space $P \cong \mathbb{H}$ and its orthogonal complement $P^\perp \cong \mathbb{R}$. With respect to this splitting, we can write the 4-dimensional unit sphere in the following fashion:

$$S^4 = \left\{(\mathbf{x}, y) \in P \oplus P^\perp : |\mathbf{x}|^2 + |y|^2 = 1\right\}.$$

Now, for all $(\mathbf{x}, y) \in S^4$ there exists a unique $\alpha \in [-\pi/2, \pi/2]$ such that

$$\mathbf{x} = \cos\alpha \mathbf{u}, \qquad y = \sin\alpha,$$

for some $\mathbf{u} \in S^3$. Note that $\mathbf{u}$ is uniquely determined when $\alpha \neq \pm\pi/2$. Essentially, we are writing S^4 as a 1-parameter family of S^3s that are collapsing to a point on each end of the parametrization.

Let $\{\partial_1, \partial_2, \partial_3\}$ be the standard left-invariant orthonormal frame on $S^3 \cong \mathrm{Sp}(1)$. Considering this frame in the description of S^4 above, we deduce that

$$f_0 := \partial_\alpha, \quad f_1 := \frac{\partial_1}{\cos\alpha}, \quad f_2 := \frac{\partial_2}{\cos\alpha}, \quad f_3 := \frac{\partial_3}{\cos\alpha},$$

is an oriented orthonormal frame of $S^4 \setminus \{\alpha = \pm\pi/2\}$. The dual coframe is:

$$b_0 := d\alpha; \quad b_1 := \cos\alpha\sigma_1; \quad b_2 := \cos\alpha\sigma_2; \quad b_3 := \cos\alpha\sigma_3, \tag{4.3.1}$$

where $\{\sigma_i\}_{i=1}^3$ is the dual coframe of $\{\partial_i\}_{i=1}^3$ in S^3, which is well-known to satisfy:

$$d\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\end{pmatrix} = 2\begin{pmatrix}\sigma_2\wedge\sigma_3\\ \sigma_3\wedge\sigma_1\\ \sigma_1\wedge\sigma_2\end{pmatrix}. \tag{4.3.2}$$

We deduce that the round metric on the unit sphere S^4 can be written as:

$$g_{S^4} = d\alpha^2 + \cos^2\alpha g_{S^3},$$

and the volume form is:

$$\mathrm{vol}_{S^4} = \cos^3\alpha d\alpha \wedge \mathrm{vol}_{S^3},$$

where $g_{S^3} = \sigma_1^2 + \sigma_2^2 + \sigma_3^2$ and $\mathrm{vol}_{S^3} = \sigma_1\wedge\sigma_2\wedge\sigma_3$.

4.3.2 Horizontal and the vertical space

Exactly as in Section 4.2.2 we can compute the connection 1-forms ρ_i for $i = 1, 2, 3$ with respect to the coframe we have constructed. Indeed, a straightforward computation involving Eq. (2.3.1), Eq. (4.3.1) and Eq. (4.3.2) implies that $\rho_i = l\sigma_i$ for all $i = 1, 2, 3$, where

$$l := \frac{\sin\alpha - 1}{2}.$$

Hence, we can deduce from Eq. (2.3.2) that the vertical 1-forms in these coordinates are:

$$\begin{aligned} \xi_0 &= da_0 + l(a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3), & \xi_1 &= da_1 + l(-a_0\sigma_1 - a_2\sigma_3 + a_3\sigma_2), \\ \xi_2 &= da_2 + l(-a_0\sigma_2 + a_1\sigma_3 - a_3\sigma_1), & \xi_3 &= da_3 + l(-a_0\sigma_3 - a_1\sigma_2 + a_2\sigma_1). \end{aligned} \tag{4.3.3}$$

4.3.3 SU(2)-action

Given the splitting of $\mathbb{R}^5$ into $P \cong \mathbb{H}$ and its orthogonal complement $P^\perp$, we can consider $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$ acting via left multiplication on P and trivially on $P^\perp$. Equivalently, we are considering $\mathrm{Sp}(1) \cong \mathrm{Sp}(P) \times \mathrm{Id}_{P^\perp} \subset \mathrm{SO}(5)$. Being a subgroup of SO(5), the action descends to the unit sphere S^4.

We first consider $\alpha \neq -\pi/2$, where we trivialize $S^4 \setminus \{\text{south pole}\}$ using homogeneous quaternionic coordinates on $\mathbb{HP}^1 \cong S^4$. In this chart, diffeomorphic to $\mathbb{H}$, the action is given by standard left multiplication.

We extend the action on S^4 to the tangent bundle of S^4 via the differential. In this trivialization, $\mathbb{H} \times \mathbb{H}$, the action is given by left-multiplication on both factors. Hence, if we pick the trivialization of $P_{\mathrm{SO}(4)}$ induced by $\{1, i, j, k\}$, the action of $p \in \mathrm{Sp}(1)$ maps the element $(x, \mathrm{Id}_{\mathrm{SO}(4)}) \in \mathbb{H} \times \mathrm{SO}(4)$ to $(p \cdot x, \tilde{p})$, where

$$\tilde{p} = \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix}.$$

By the simply-connectedness of $\mathrm{Sp}(1) \cong \mathrm{Spin}(3)$, we can lift the action to the spin structure $P_{\mathrm{Spin}(4)}$ of S^4. Using a similar diagram to (Eq. (4.2.4)) and the fact that the lift of $\tilde{p}$ is $(p, \mathrm{Id}_{\mathrm{Sp}(1)}) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1)$, we can show that in the trivialization of $P_{\mathrm{Spin}(4)}$, $\mathbb{H} \times \mathrm{Sp}(1) \times \mathrm{Sp}(1)$, the element $(x, (\mathrm{Id}_{\mathrm{Sp}(1)}, \mathrm{Id}_{\mathrm{Sp}(1)}))$ is mapped to $(p \cdot x, (p, \mathrm{Id}_{\mathrm{Sp}(1)}))$.

As in Section 4.2, this passes to the quotient space: $\not{S}_-(S^4)$, and, in the induced trivialization, $\mathbb{H} \times \mathbb{H}$, the action of Sp(1) is only given by left multiplication on the first factor by definition of μ_-.

A similar argument works for the other chart of $\mathbb{HP}^1$. However, the left multiplication becomes right multiplication of the conjugate, and the lift of the new $\tilde{p}$ is $(\mathrm{Id}_{\mathrm{Sp}(1)}, p)$. It follows that $\mathrm{Sp}(1)$ acts on the fibre over the south pole as it acts on $\mathbb{H}$.

In particular, we proved the following lemma.

Lemma 4.3.2. *The orbits of the* $\mathrm{SU}(2)$*-action on* $\not{S}_-(S^4)$ *are given in Table 4.2.*

α	a	Orbit
$\neq \pm\frac{\pi}{2}$		S^3
$= -\frac{\pi}{2}$	$\neq 0$	S^3
$= -\frac{\pi}{2}$	$= 0$	Point
$= \frac{\pi}{2}$		Point

Table 4.2: Spin(3) Orbits

When $\alpha \neq \pm\pi/2$ we can use the orthonormal frame of Section 4.3.1. Obviously, it is invariant under the action. Hence, in the induced trivialization of $\not{S}_-(S^4)$, $\mathrm{Sp}(1)$ acts only on the component of the basis. In particular, it follows that $\{\sigma_1, \sigma_2, \sigma_3\}$ is a coframe on the orbits of the $\mathrm{SU}(2)$-action, and, $\{\partial_1, \partial_2, \partial_3\}$ is the relative frame. Observe that we are working on the coframe $\{d\alpha, \sigma_1, \sigma_2, \sigma_3, da_0, da_1, da_2, da_3\}$.

4.3.4 Choice of frame and the $\mathrm{Spin}(7)$ geometry in the adapted coordinates

Since the considered $\mathrm{SU}(2)$-action only moves the base of the vector bundle $\not{S}_-(S^4)$ in the trivialization of Section 4.3.1, it is natural to use: $\{d\alpha, \sigma_1, \sigma_2, \sigma_3, \xi_0, \xi_1, \xi_2, \xi_3\}$. The metrics g_c and the Cayley forms Φ_c admit a nice formula with respect to this coframe. Recall that we are working on the chart $\mathcal{U} := \not{S}_-(S^4) \setminus \{\alpha = \pm\pi/2\}$.

Proposition 4.3.3. *Given $c \geq 0$, the Riemannian metric g_c, in the coframe considered in this subsection, satisfies:*

$$g_c = 5(c + r^2)^{3/5}\left(d\alpha^2 + \cos^2\alpha\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)\right) + 4(c + r^2)^{-2/5}\left(\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2\right), \tag{4.3.4}$$

where $r^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2$.

Given $c \geq 0$, the Cayley form Φ_c, in the coframe considered in this subsection, satisfies:

$$\begin{aligned}\Phi_c =& 16(c + r^2)^{-4/5}\xi_0 \wedge \xi_1 \wedge \xi_2 \wedge \xi_3 + 25(c + r^2)^{6/5}\cos^3\alpha d\alpha \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3 \\ &+ 20(c + r^2)^{1/5}\cos\alpha\left(\sum_{i=1}^{3}(\xi_0 \wedge \xi_i - \xi_j \wedge \xi_k) \wedge (d\alpha \wedge \sigma_i - \cos\alpha\sigma_j \wedge \sigma_k)\right),\end{aligned} \tag{4.3.5}$$

where $r^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2$.

Proof. It follows immediately from Eq. (2.3.3), Eq. (2.3.4) and the choice of the coframe. $\square$

If we denote by $\{e_\alpha, e_1, e_2, e_3, e_{\xi_0}, e_{\xi_1}, e_{\xi_2}, e_{\xi_3}\}$ the frame dual to $\{d\alpha, \sigma_1, \sigma_2, \sigma_3, \xi_0, \xi_1, \xi_2, \xi_3\}$, it is straightforward to relate these vectors to $\partial_\alpha, \partial_1, \partial_2, \partial_3, \partial_{a_0}, \partial_{a_1}, \partial_{a_2}, \partial_{a_3}$.

Lemma 4.3.4. *The dual frame* $\{e_\alpha, e_1, e_2, e_3, e_{\xi_0}, e_{\xi_1}, e_{\xi_2}, e_{\xi_3}\}$ *satisfies:*

$$\begin{aligned}
e_\alpha &= \partial_\alpha;\\
e_1 &= \partial_1 + l\left(-a_1\partial_{a_0} + a_0\partial_{a_1} + a_3\partial_{a_2} - a_2\partial_{a_3}\right);\\
e_2 &= \partial_2 + l\left(-a_2\partial_{a_0} - a_3\partial_{a_1} + a_0\partial_{a_2} + a_1\partial_{a_3}\right);\\
e_3 &= \partial_3 + l\left(-a_3\partial_{a_0} + a_2\partial_{a_1} - a_1\partial_{a_2} + a_0\partial_{a_3}\right);\\
e_{\xi_i} &= \partial_{a_i} \quad \forall i = 0, 1, 2, 3,
\end{aligned}$$

where l is as defined in Section 4.3.2.

Proof. It is straightforward from the definition of dual frame and Eq. (4.3.3). $\square$

4.3.5 Cayley condition

Analogously to the case carried out in Section 4.2, the generic orbits of the considered SU(2)-action are 3-dimensional (see Lemma 4.3.2). Hence, it is sensible to look for invariant Cayley submanifolds. To this purpose, we assume that the submanifold N consists of a 1-parameter family of 3-dimensional SU(2)-orbits in M. In particular, the coordinates that do not describe the orbits, i.e. a_0, a_1, a_2, a_3 and α, need to be functions of a parameter τ. This means that we can write:

$$N = \{((\cos\alpha(\tau)\mathbf{u}, \sin\alpha(\tau)), (a_0(\tau), a_1(\tau), a_2(\tau), a_3(\tau))) : |u| = 1, \tau \in (-\epsilon, \epsilon)\}. \qquad (4.3.6)$$

The tangent space is spanned by $\{\partial_1, \partial_2, \partial_3, \dot{\alpha}\partial_\alpha + \sum_{i=0}^{3} \dot{a}_i\partial_{a_i}\}$, where the dots denote the derivatives with respect to τ. The condition under which N is Cayley becomes a system of ODEs.

Proposition 4.3.5. *Let N be an* SU(2)*-invariant submanifold as described at the beginning of this subsection. Then, N is Cayley in the chart $\mathcal{U}$ if and only if the following* *system of ODEs is satisfied:*

$$\begin{cases} \dot{a}_0 a_1 - \dot{a}_1 a_0 - \dot{a}_2 a_3 + \dot{a}_3 a_2 = 0 \\ \dot{a}_0 a_2 + \dot{a}_1 a_3 - \dot{a}_2 a_0 - \dot{a}_3 a_1 = 0 \\ \dot{a}_0 a_3 - \dot{a}_1 a_2 + \dot{a}_2 a_1 - \dot{a}_3 a_0 = 0 \\ \cos\alpha(-f\cos^2\alpha + 3l^2 g r^2)\dot{a}_0 - l(l^2 g r^2 - 3f\cos^2\alpha) a_0 \dot{\alpha} = 0 \\ \cos\alpha(-f\cos^2\alpha + 3l^2 g r^2)\dot{a}_1 - l(l^2 g r^2 - 3f\cos^2\alpha) a_1 \dot{\alpha} = 0 \\ \cos\alpha(-f\cos^2\alpha + 3l^2 g r^2)\dot{a}_2 - l(l^2 g r^2 - 3f\cos^2\alpha) a_2 \dot{\alpha} = 0 \\ \cos\alpha(-f\cos^2\alpha + 3l^2 g r^2)\dot{a}_3 - l(l^2 g r^2 - 3f\cos^2\alpha) a_3 \dot{\alpha} = 0 \end{cases}$$

where $r^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2$, $l = (\sin\alpha - 1)/2$, $f = 5(c + r^2)^{3/5}$ *and* $g = 4(c + r^2)^{-2/5}$.

Proof. We first write the tangent space of N, which is spanned by $\{\partial_1, \partial_2, \partial_3, \dot{\alpha}\partial_\alpha + \sum_{i=0}^3 \dot{a}_i \partial_{a_i}\}$, in terms of the frame $\{e_\alpha, e_1, e_2, e_3, e_{\xi_0}, e_{\xi_1}, e_{\xi_2}, e_{\xi_3}\}$. This can be easily done using Lemma 4.3.4. Through a long computation analogous to the one carried out in Section 4.2.7.1, we can apply Proposition 2.3.9 to this case, and we obtain the system of ODEs. $\square$

Remark 4.3.6. It is interesting to point out that, exactly as in the $\mathrm{SO}(3) \times \mathrm{Id}_2$ case (see Lemma 4.2.16), the projection π_7 of Proposition 2.3.9 will just be the identity in the proof of Proposition 4.3.5.

4.3.6 Cayley fibration

In the previous section we found the condition that makes N, SU(2)-invariant submanifold, Cayley. This consists of a system of ODEs, which will characterize the desired Cayley fibration.

Harvey and Lawson local existence and uniqueness theorem implies that any SU(2)-invariant Cayley can meet the zero section only when $\alpha = \pm\pi/2$, i.e. outside of $\mathcal{U}$. Otherwise, the zero section of $\$_-(S^4)$, which is Cayley, would intersect such an N in a 3-dimensional submanifold, contradicting Harvey and Lawson theorem. It follows that the initial value of one of the a_is is different from zero. We take $a_0(0) \neq 0$, as the other cases will follow similarly. Now, it is straightforward to notice that:

$$a_1 = \frac{a_1(0)}{a_0(0)} a_0; \quad a_2 = \frac{a_2(0)}{a_0(0)} a_0; \quad a_3 = \frac{a_3(0)}{a_0(0)} a_0; \tag{4.3.7}$$

solves the first 3 equations of the system given in Proposition 4.3.5. Moreover, it also reduces the remaining equations to the ODE:

$$\cos\alpha(-f\cos^2\alpha + 3l^2 g r^2)\dot{a}_0 - l(l^2 g r^2 - 3f\cos^2\alpha) a_0 \dot{\alpha} = 0,$$

where, as usual, $r^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2$, $l = (\sin\alpha - 1)/2$, $f = 5(c+r^2)^{3/5}$ and $g = 4(c+r^2)^{-2/5}$. As Eq. (4.3.7) implies that $a_0 = p^{-1}r$, where p is the positive real number satisfying $p^2 = 1 + \sum_{i=1}^{3}(a_i(0)/a_0(0))^2$, we can rewrite the previous ODE as:

$$\cos\alpha(-f\cos^2\alpha + 3l^2gr^2)\dot{r} - l(l^2gr^2 - 3f\cos^2\alpha)r\dot{\alpha} = 0. \tag{4.3.8}$$

Remark 4.3.7. It is easy to verify that Eq. (4.3.8) is not in exact form. Hence, it cannot be easily integrated. It is a non-trivial open task to verify whether, possibly up to change of coordinates, Eq. (4.3.8) can be integrated in closed form.

In order to understand the SU(2)-invariant Cayley fibrations, we analyse the ODE given in Eq. (4.3.8). First, we deduce the sign of $f_1 := \cos\alpha(-f\cos^2\alpha + 3l^2gr^2)$. If we let

$$\alpha_c(r) := \arcsin\left(-\frac{2r^2+5c}{8r^2+5c}\right),$$

it easy to verify that f_1 is positive on the left of α_c for $(\alpha, r) \in (-\pi/2, \pi/2) \times \mathbb{R}^+$, and negative otherwise. Moreover, f_1 vanishes along the 3 curves $\alpha_c, \alpha = \pm\pi/2$; there, f_1 changes sign. Note that $\alpha_c \to \arcsin(-1/4)$ as $r \to \infty$.

Now, we consider $f_2 := l(l^2gr^2 - 3f\cos^2\alpha)r$. Letting

$$\beta_c(r) := \arcsin\left(-\frac{14r^2+15c}{16r^2+15c}\right),$$

then, f_2 is positive on the right of β_c for $(\alpha, r) \in (-\pi/2, \pi/2) \times \mathbb{R}^+$, and it is negative otherwise. Obviously, f_2 vanishes along the curve β_c and the vertical line $\alpha = \pi/2$. Note that $\beta_c \to \arcsin(7/8)$ as $r \to \infty$. The last key observation is that f_2/f_1 tends to zero as α tends to $\pi/2$.

Putting what said so far together, and observing that $\beta_c(r) < \alpha_c(r)$ for all $r > 0$, we can draw the flow lines for Eq. (4.3.8) (see Fig. 4.5). Finally, we can use these to deduce the form of the solutions from standard arguments (see Fig. 4.6).

The conical version. We consider the easier conical case first. From a topological point of view, it is obvious that the red and green Cayleys of Fig. 4.6 (B) are homeomorphic to $S^3 \times \mathbb{R}$. As the the group action becomes trivial on $\alpha = \pi/2$, the topology of the fibres in blue cannot be recovered from the picture. However, it will be clear from the asymptotic analysis that these are smooth topological $\mathbb{R}^4$s. As a consequence, we have constructed a Cayley fibration on the chart $\mathcal{U} \cap M_0$, which extends to the whole M_0 by continuity (i.e. we complete the Cayleys in blue and we add the whole π_0-fibre at $\alpha = -\pi/2$). On M_0 the Cayley fibration remains a fibration in the classical sense. A reasoning similar to the one of Section 4.2 shows that the parametrizing space $\mathcal{B}$ of the Cayley fibration is $\mathbb{R}^4$.

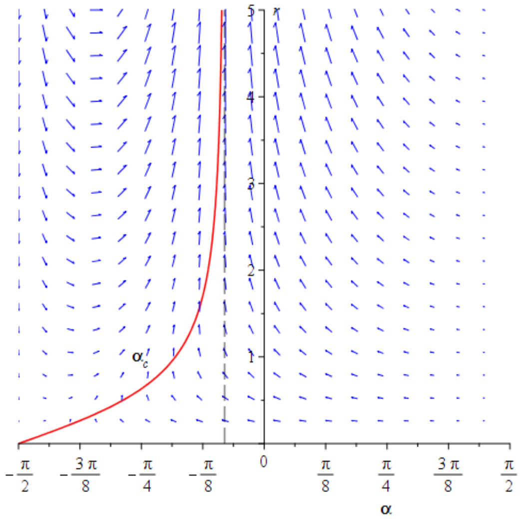
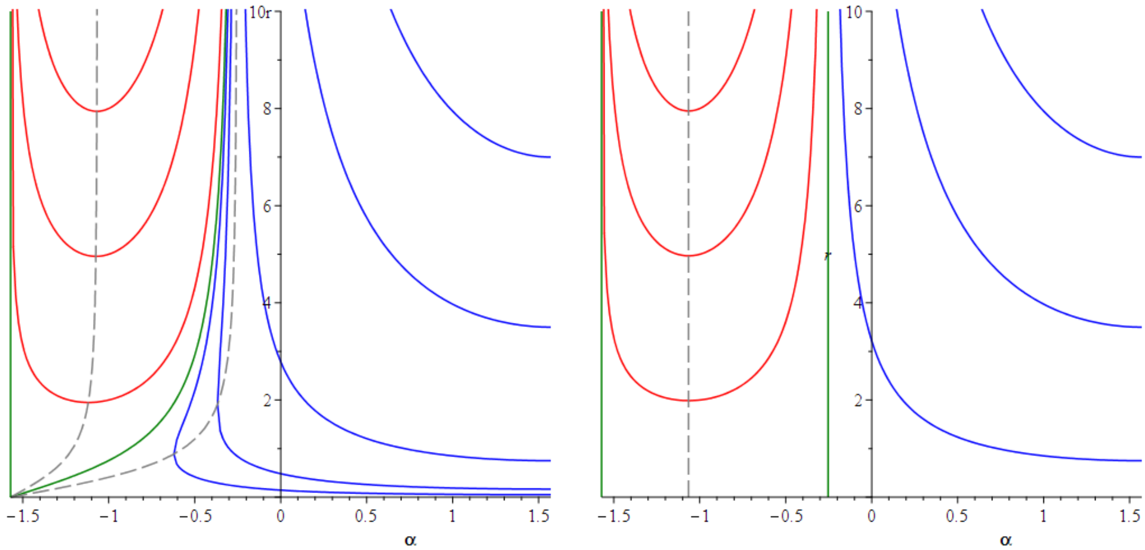


Figure 4.5: Flow lines for $Eq.$ (4.3.8).



(a) generic case

(b) conical case

Figure 4.6: Solutions of $Eq.$ (4.3.8).

The smooth version. Now, we deal with the generic case $c > 0$. As above, the topology of the red Cayleys of Fig. 4.6 (A) is $S^3 \times \mathbb{R}$; the blue ones have topology $\mathbb{R}^4$. In the latter, we use the same asymptotic analysis argument of the conical case. Finally, the submanifolds in green are smooth topological $\mathbb{R}^4$s. As usual, we extend the Cayley fibration on $\mathcal{U}$ to the whole M by continuity (i.e. we add the whole π_c-fibre over $\alpha = -\pi/2$, we complete the Cayleys in blue and green, and we add the zero section S^4). Observe that the zero section, the π_c-fibre over $\alpha = -\pi/2$ and the green Cayleys all intersect in a point p. It follows that the M' given in Definition 3.1.5 is equal to $M \setminus \{p\}$. Once again, a reasoning similar to the one of Section 4.2 shows that the parametrizing space $\mathcal{B}$ of the Cayley fibration is S^4.

The smoothness of the fibres (the asymptotic analysis as $r \to r_0 \geq 0$) In this subsection, we study the smoothness of the fibres. This is trivial as long as the submanifolds are contained in $\mathcal{U}$; hence, the Cayleys of topology $S^3 \times \mathbb{R}$ are smooth, and we only need to check the others at the points where they meet $\partial\mathcal{U}$. To this purpose, we carry out a asymptotic analysis similar to the one of Section 4.2.

As a first step, we restrict the metric g_c to N. Combining *Eq.* (4.3.4) together with *Eq.* (4.3.7) and its consequence $a_0 = p^{-1}r$ for p positive real number satisfying $p^2 = 1 + \sum_{i=1}^3 (a_i(0)/a_0(0))^2$, we can write the restriction as follows:

$$\begin{aligned} g_c\big|_N =& \left(5(c+r^2)^{3/5}\cos^2\alpha + 4(c+r^2)^{-2/5} l^2 r^2\right)\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right) \\ &+4(c+r^2)^{-2/5}dr^2 + 5(c+r^2)^{3/5}d\alpha^2, \end{aligned} \tag{4.3.9}$$

where α and r are related by the differential equation: Eq. (4.3.8) and, as usual, $l = (\sin\alpha - 1)/2$.

$(r \to r_0)$. Recall that $f_2/f_1 \to 0$ as $\alpha \to \pi/2$. Therefore, the Cayleys around $\alpha = \pi/2$ are asymptotic to the horizontal line $\alpha = r_0$ for some constant $r_0 \geq 0$. By Eq. (4.3.9), the metric in this first order linear approximation becomes:

$$g_c\big|_N \sim 5(c + r_0^2)^{3/5}\left(d(\alpha - \pi/2)^2 + (\alpha - \pi/2)^2\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right)\right).$$

In this way, we have proved that near $\alpha = \pi/2$ every Cayley we have constructed is smooth. Moreover, we can also deduce that the blue Cayleys of Figure Fig. 4.6 are topologically $\mathbb{R}^4$s.

$(r \to 0)$. Finally, we need to check whether the remaining Cayleys of topology $\mathbb{R}^4$ are smooth or not. In this situation we can approximate them near $\alpha = -\pi/2$ with the submanifold associated to the line:

$$\alpha = Ar - \frac{\pi}{2},$$

where A is some positive constant (as the lines corresponding to the Cayleys live between α_c and β_c). The metric in the linear approximation is asymptotic to:

$$g_c\big|_N \sim c^{-2/5}(5cA^2+4)\left(dr^2+r^2\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right)\right),$$

hence, we conclude that these submanifolds are smooth as well.

The main theorems Putting all these results together we obtain the following theorems.

Theorem 4.3.8 (T. [71]; Generic case)**.** *Let (M,Φ_c) be the Bryant–Salamon manifold constructed over the round sphere S^4 for some $c>0$, and let* $\mathrm{SU}(2)$ *act on M as in Section 4.3.3. Then, M admits an* $\mathrm{SU}(2)$*-invariant Cayley fibration parametrized by $\mathcal{B}\cong S^4$. The fibres are topologically $S^3\times\mathbb{R}$, S^4 and $\mathbb{R}^4$. All the Cayleys are smooth. There is only one point where multiple fibres intersect. This point lies in the zero section of $\$_-(S^4)$, and there are $S^3\sqcup\{\text{two points}\}$ Cayleys passing through it.*

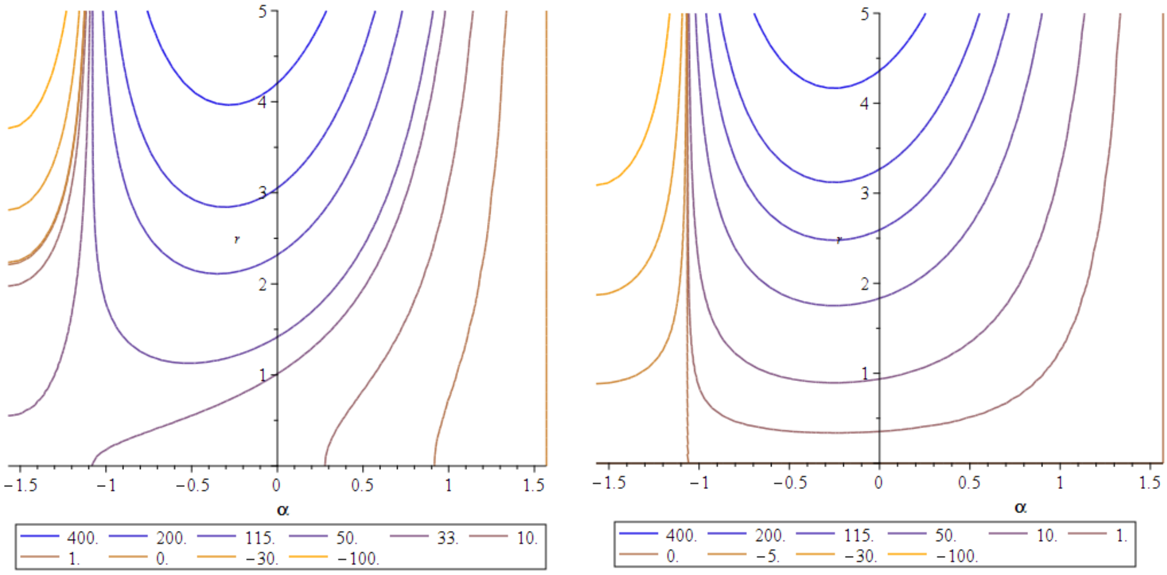
Theorem 4.3.9 (T. [71]; Conical case)**.** *Let (M_0,Φ_0) be the conical Bryant–Salamon manifold constructed over the round sphere S^4, and let* $\mathrm{SU}(2)$ *act on M_0 as in Section 4.3.3. Then, M_0 admits an* $\mathrm{SU}(2)$*-invariant Cayley fibration parametrized by $\mathcal{B}\cong\mathbb{R}^4$. The fibres are topologically $S^3\times\mathbb{R}$ or $\mathbb{R}^4$ and are all smooth. Moreover, as these do not intersect, the* $\mathrm{SU}(2)$*-invariant Cayley fibration is a fibration in the usual differential geometric sense with fibres Cayley submanifolds.*

Remark 4.3.10. Blowing-up at the north pole, it is easy to see that the Cayley fibration becomes trivial in the limit.

Remark 4.3.11. As in the previous section, we are able to compute the multi-moment maps relative to this action explicitly. Indeed, this is:

$$\nu_c := \frac{5}{6}(r^2-5c)(c+r^2)^{1/5}(\sin\alpha-1)^3-\frac{25}{2}(c+r^2)^{6/5}\cos^2\alpha(\sin\alpha-1).$$

In order to provide an idea on how the multi-moment maps behave, we draw the level sets of ν_1 and ν_0 (see Figure Fig. 4.7).



(a) Level sets of ν_1 (b) Level sets of ν_0

Figure 4.7: Level sets of the multi-moment map in the generic and conical case

Asymptotic geometry as $r \to \infty$. The first observation we need to make is that there are only two asymptotic behaviours for the Cayleys constructed in Theorem 4.3.8 and in Theorem Theorem 4.3.9: one corresponding to $\alpha \sim -\pi/2$ and the other to $\alpha \sim \arcsin(-1/4)$. In both cases, we can use Eq. (4.3.9) to obtain the asymptotic cone, which is:

$$g_c\big|_N \sim ds^2 + \frac{9}{25}s^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2),$$

for $\alpha \sim \pi/2$, and it is

$$g_c\big|_N \sim ds^2 + \frac{9}{16}s^2(\sigma_1^2 + \sigma_2^2 + \sigma_3^2),$$

for $\alpha \sim \arcsin(-1/4)$, where $s := (10/3)r^{3/5}$.

4.4 Cayley fibration invariant under the lift of the $\mathrm{SO}(3)$ irreducible action on S^4

Differently from the other actions, where it was possible to describe the invariant fibrations explicitly, this is not the case when $\mathrm{SO}(3)$ acts irreducibly on S^4. Indeed, irreducibility implies that there is no simple frame on S^4 compatible with the group action. Hence, the Cayley condition, and consequently the associated ODEs, will become extremely complicated.

Moreover, the analogous action on the flat Spin(7) space and on the Bryant-Salamon G_2 manifold $\Lambda^2(T^*S^4)$ was studied by Lotay [58, Subsection 5.3.3] and Kawai [51], respectively. In both cases, the defining ODEs for Cayley submanifolds and coassociative submanifolds were too complicated to be explicitly solved.

Chapter 5

Calibrated geometry in G_2 manifolds with $\mathbb{T}^2 \times \mathrm{SU}(2)$-symmetry

This chapter, based on the joint work with Aslan [6], is focused on G_2 manifolds admitting a $\mathbb{T}^2 \times \mathrm{SU}(2)$-action, which we assume to be of cohomogeneity-two. In particular, we provide a local characterization of these manifolds, that reduces the torsion-free condition to two nested systems of ODEs, and we consider the following natural families of calibrated submanifolds in them: $\mathbb{T}^2 \times \mathrm{Id}_{\mathrm{SU}(2)}$-invariant associative submanifolds, $\mathbb{T}^2 \times S^1$-invariant coassociative submanifolds for some $S^1 < \mathrm{SU}(2)$ and $\mathrm{Id}_{\mathbb{T}^2} \times \mathrm{SU}(2)$-invariant coassociative submanifolds.

As in Chapter 4, the invariance reduces the problem of findings such objects into a system of ODEs in the appropriate orbit space. However, since we are working with an enhanced symmetry, we can project the solutions of such ODEs to the quotient of the whole group, $\mathbb{T}^2 \times \mathrm{SU}(2)$. It turns out that, on this 2-dimensional quotient, the $\mathbb{T}^2$-invariant associatives and the $\mathbb{T}^3$-invariant coassociatives correspond to the level sets of some combination of the associated multi-moment maps, which act as local coordinates for the surface. The $\mathrm{SU}(2)$-invariant coassociatives, when they exist, correspond to the integral curves of a nowhere vanishing vector field, once again induced from a multi-moment map.

Moreover, we show that $\mathbb{T}^2$-invariant associatives and $\mathrm{SU}(2)$-invariant coassociatives are smooth, while the $\mathbb{T}^3$-invariant coassociatives develop singularities with one tangent cone diffeomorphic to a line times the Harvey–Lawson cone (see [37]).

We apply our discussion to the flat space, to the manifolds constructed by Foscolo–Haskins–Nordström in [32] and on the Bryant–Salamon manifolds of topology $S^3 \times \mathbb{R}^4$. In particular, we obtain new examples of $\mathbb{T}^2$-invariant associatives in the latter two cases.

5.1 G_2 manifolds with $\mathbb{T}^2 \times \mathrm{SU}(2)$-symmetry

In this section, we consider a G_2 manifold (M, φ) with a structure-preserving $\mathbb{T}^2 \times \mathrm{SU}(2)$-action of cohomogeneity two, i.e. the maximal dimension achieved by the orbits is 5.

5.1.1 $\mathbb{T}^2 \times \mathrm{SU}(2)$-symmetry

To understand the action of $\mathbb{T}^2 \times \mathrm{SU}(2)$ on M, let Γ be the kernel of the homomorphism $\mathbb{T}^2 \times \mathrm{SU}(2) \to \mathrm{Aut}(M)$, which is discrete by assumption. Once we rewrite it as $\Gamma = \{(a_i, b_i) \in \mathbb{T}^2 \times \mathrm{SU}(2) : i \in I\}$, we define $\Gamma_1 := \{a \in \mathbb{T}^2 : (a, \mathrm{Id}_{\mathrm{SU}(2)}) \in \Gamma\}$ and $\Gamma_2 := \{b \in \mathrm{SU}(2) : (\mathrm{Id}_{\mathbb{T}^2}, b) \in \Gamma\}$, which are subgroups of $\mathbb{T}^2$ and $\mathrm{SU}(2)$ respectively.

Consider the $\mathbb{T}^2$ action on M given by $\mathbb{T}^2 \times \mathrm{Id}_{\mathrm{SU}(2)} \subset \mathbb{T}^2 \times \mathrm{SU}(2)$. Since

$$\Gamma_1 \times \mathrm{Id}_{\mathrm{SU}(2)} = (\mathbb{T}^2 \times \mathrm{Id}_{\mathrm{SU}(2)}) \cap \Gamma,$$

we see that the action of $\mathbb{T}^2/\Gamma_1$ is effective, and, as $\mathbb{T}^2/\Gamma_1$ is diffeomorphic to $\mathbb{T}^2$, we can assume, without loss of generality, that Γ_1 is trivial and that the action of $\mathbb{T}^2 \cong \mathbb{T}^2 \times \mathrm{Id}_{\mathrm{SU}(2)}$ is effective. We denote by $\mathcal{S}$ the singular set of this action, i.e. the complement of the principal set with respect to this action.

Analogously, we have an $\mathrm{SU}(2)$-action on M given by $\mathrm{SU}(2) \cong \mathrm{Id}_{\mathbb{T}^2} \times \mathrm{SU}(2) \subset \mathbb{T}^2 \times \mathrm{SU}(2)$, which induces an effective action of $\mathrm{SU}(2)/\Gamma_2$. The singular set of this action is denoted by $\tilde{\mathcal{S}}$.

Remark 5.1.1. Observe that Γ does not need to be equal to $\Gamma_1 \times \Gamma_2$. For instance, if $\Gamma = \{\pm(1,1)\}$, then, Γ_1 and Γ_2 are trivial.

Now, we show that Γ is in the center of $\mathbb{T}^2 \times \mathrm{SU}(2)$: $Z(\mathbb{T}^2 \times \mathrm{SU}(2)) = \mathbb{T}^2 \times \{\pm 1\}$.

Lemma 5.1.2. *Let $x \in M$ be such that the stabilizer $(\mathbb{T}^2 \times \mathrm{SU}(2))_x$ is discrete. Then, the stabilizer is a subgroup of the center $Z(\mathbb{T}^2 \times \mathrm{SU}(2))$.*

Proof. We show that the adjoint representation of $(\mathbb{T}^2 \times \mathrm{SU}(2))_x$ on $\mathfrak{t}^2 \oplus \mathfrak{su}(2)$ is trivial, which implies the statement by naturality of the exponential map.

Let $N = \nu_x$ be the normal space at x of the $\mathbb{T}^2 \times \mathrm{SU}(2)$-orbit, whose tangent space is identified with $\mathfrak{t}^2 \oplus \mathfrak{su}(2)$ in the usual manner. Then, the representation of $(\mathbb{T}^2 \times \mathrm{SU}(2))_x$ on T_xM splits as

$$T_xM = \mathfrak{t}^2 \oplus \mathfrak{su}(2) \oplus N, \tag{5.1.1}$$

and coincides with the adjoint representation on the $\mathfrak{t}^2 \oplus \mathfrak{su}(2)$ part. Being abelian, the action on $\mathfrak{t}^2$ is trivial and the same holds for the cross product of the $\mathfrak{t}^2$-generators, which spans a linear subspace N_1 of N by Eq. (5.1.6). Note that we used that the action of $(\mathbb{T}^2 \times \mathrm{SU}(2))_x$ preserves the G_2-structure. Denote by N_2 the orthogonal complement of N_1 in N, which is invariant under the action. Being an isometry, every element $g \in (\mathbb{T}^2 \times \mathrm{SU}(2))_x$ acts on N_2 by multiplication of λ_g, where $\lambda_g \in \{-1, +1\}$.

Finally, we show that λ_g cannot be -1. In order to do so, we consider the map $(\mathfrak{t}^2 \oplus N_1) \otimes N_2 \to \mathfrak{su}(2)$ which is the composition of the cross product and the projection onto the $\mathfrak{su}(2)$ component in the splitting given by Eq. (5.1.1). Since $\mathfrak{t}^2 \oplus N_1$ is an associative subspace, this map is an isomorphism of representations. Hence, g acts on $\mathfrak{su}(2)$ by multiplication of λ_g. We conclude the proof because there is no element in $\mathbb{T}^2 \times \mathrm{SU}(2)$ whose adjoint action on $\mathfrak{su}(2)$ is multiplication by -1. $\square$

Corollary 5.1.3. *Since* $\mathbb{T}^2 \times \mathrm{SU}(2)$ *acts with cohomogeneity two,* Γ *is in the centre of* $\mathbb{T}^2 \times \mathrm{SU}(2)$*. Hence,* $\mathrm{SU}(2)/\Gamma_2$ *is either* $\mathrm{SU}(2)$ *or* $\mathrm{SO}(3)$*.*

Corollary 5.1.4. *The principal stabilizer of* $(\mathbb{T}^2 \times \mathrm{SU}(2))/\Gamma$ *is trivial.*

Proof. As a consequence of Lemma 5.1.2, all principal stabilizer subgroups are not only conjugate, but equal to each other. Since the action is effective after the quotient, the principal stabilizer needs to be trivial. $\square$

From now on, we consider the action of $G := (\mathbb{T}^2 \times \mathrm{SU}(2))/\Gamma \subset \mathrm{Aut}(M, \varphi)$, and we denote by M_P its principal set. This is going to greatly simplify our arguments, indeed, the G-action is effective and with trivial principal stabilizer.

We will make use of two additional actions induced from the original $\mathbb{T}^2 \times \mathrm{SU}(2)$. Let $\tilde{\Gamma}_1 := \{a_i : (a_i, b_i) \in \Gamma\}$ and let $\tilde{\Gamma}_2 := \{b_i : (a_i, b_i) \in \Gamma\}$, which is either trivial or $\{\pm 1\}$ by Corollary 5.1.3. We state the following lemma without proof.

Lemma 5.1.5. *Let* $\mathbb{T}^2 \cong \mathbb{T}^2 \times \mathrm{Id}_{\mathrm{SU}(2)}$ *acting on* M*. Then, there exists an induced action of* $G^{\mathbb{T}^2} := \mathbb{T}^2/\tilde{\Gamma}_1$ *on* $M_P/(\mathrm{SU}(2)/\Gamma_2)$ *which is free. In particular,* $M_P/(\mathrm{SU}(2)/\Gamma_2)$ *becomes a principal* $G^{\mathbb{T}^2}$*-bundle over* $B := M_P/G$*. Similarly, there exists a* $G^{\mathrm{SU}(2)} := \mathrm{SU}(2)/\tilde{\Gamma}_2$ *action induced by* $\mathrm{SU}(2) \cong \mathrm{Id}_{\mathbb{T}^2} \times \mathrm{SU}(2)$ *on* $M_P/\mathbb{T}^2$ *which is free. As before,* $M_P/\mathbb{T}^2$ *becomes a principal* $G^{\mathrm{SU}(2)}$*-bundle over* B*.*

The various group quotient are summarised in the following diagram:

$$\begin{tikzcd}
& M_P \arrow[dl, "/(\mathrm{SU}(2)/\Gamma_2)"'] \arrow[dr, "/\mathbb{T}^2"] \arrow[dd, "/G"] & \\
M_P/(\mathrm{SU}(2)/\Gamma_2) \arrow[dr, "/G^{\mathbb{T}^2}"'] & & M_P/\mathbb{T}^2 \arrow[dl, "/G^{\mathrm{SU}(2)}"] \\
& B &
\end{tikzcd} .$$

5.1.2 Stratification

Applying the orbit type stratification theorem and the principal orbit type theorem to our setting, where $G = (\mathbb{T}^2 \times \mathrm{SU}(2))/\Gamma$ acts effectively on M, we see that M decomposes as the union of G-orbit types, and there exists one of them which is open and dense in M. In this subsection, we study the geometry of the G-action to understand this stratification.

To simplify our notation, we fix a point $x \in M$ and denote by T the tangent space of Gx at x and by N its normal space, i.e. the orthogonal complement of T in T_xM.

In the discussion of the stratification, we will need the following lemma:

Lemma 5.1.6. *Let $\mathbb{T}^2$ be a maximal torus in G_2. Then, the representation of $\mathbb{T}^2$ on $\mathbb{R}^7$ splits as $V \oplus W_1 \oplus W_2 \oplus W_3$. Where V is 1-dimensional and each W_i is 2-dimensional. Each of $V \oplus W_i$ is an associative subspace.*

Recall that $\mathcal{S}$ is the singular set of the $\mathbb{T}^2$-action and, as a consequence of the following theorem, it is also the set where the generators of the $\mathbb{T}^2$-component are linearly dependent, i.e. there are no exceptional orbits (cfr. [64, Lemma 2.6]).

Theorem 5.1.7 (Aslan–T. [6])**.** *The dimension of the stabilizer G_x is not bigger than 4, and,*

- *if $\dim(G_x) = 0$, then, G_x is trivial, i.e. there are no exceptional orbits,*
- *if $\dim(G_x) = 1$, then, $x \notin \mathcal{S}$ and G_x is isomorphic to $\mathrm{SO}(2)$. The action of G_x on N splits as $N_1 \oplus N_2$ with $\dim(N_1) = 1, \dim(N_2) = 2$ where G_x acts trivially on N_1 and faithfully by rotations on N_2,*
- *if $\dim(G_x) = 2$, then, $x \in \mathcal{S}$ and the identity component of G_x is isomorphic to $\mathbb{T}^2$ and acts as a maximal torus in $\mathrm{U}(2)$ on N. The G-orbit Gx is an associative submanifold of M,*
- *if $\dim(G_x) = 3$, then, $x \notin \mathcal{S}$ and G_x is diffeomorpic to $\mathrm{SU}(2)$. The action of G_x on N leaves a 1-dimensional subspace $N_1 \subset N$ invariant and acts on the orthogonal complement N_2 via the standard embedding $\mathrm{SU}(2) \to \mathrm{SO}(4)$,*
- *if $\dim(G_x) = 4$, then, $x \in \mathcal{S}$ and the identity component of G_x is isomorphic to $\mathrm{U}(2)$. The action on the normal bundle N is via the embedding*

$$U(2) \to \mathrm{SU}(3), \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & \det A^{-1} \end{pmatrix}.$$

Consequently, the singular orbit set can be decomposed into $\mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$ where $\mathcal{S}_i$ is the set of points with i-dimensional stabilizer.

Proof. The first part of the proposition follows from the fact that the rank of $\mathfrak{t}^2 \oplus \mathfrak{su}(2)$ is three, while the rank of $\mathfrak{g}_2$ is two. Hence, since $G_x \subset \mathrm{G}_2$ under the identification of $(T_xM, \varphi_x) \cong (\mathbb{R}^7, \varphi_0)$, the dimension of G_x cannot be equal to 5.

By the slice theorem, a neighbourhood of Gx is equivariantly diffeomorphic to a neighbourhood of the zero section of $G \times_{G_x} N$. It follows that the representation of G_x on N is faithful. Indeed, every neighbourhood of the orbit Gx intersects M_P, on which G_x acts freely because of Corollary 5.1.4.

If $\dim(G_x) = 0$, then, an argument similar to the one used for Lemma 5.1.2 shows that G_x acts trivially on N. This means that G_x is trivial by the faithfullness of the G_x-action on N.

We now consider the case $\dim(G_x) = 1$ and $x \in \mathcal{S}$. This means that $\bar{G}_x = G_x \cap (\mathbb{T}^2 \times \mathrm{Id}_{\mathrm{SU}(2)})/\Gamma$ is not trivial and, being a subgroup of $(\mathbb{T}^2 \times \mathrm{Id}_{\mathrm{SU}(2)})/\Gamma$, it acts trivially on $T \cong \mathfrak{g}/\mathfrak{g}_x$. Since the cross-product restricted to any 4-dimensional subspace generates T_xM, we deduce that $\bar{G}_x$ acts trivially on all of T_xM. This is a contradiction as $\bar{G}_x \leq G_x$ and hence it has to act faithfully on N. We have shown that if $\dim(G_x) = 1$, then, $x \notin \mathcal{S}$. So it remains to show that G_x is isomorphic to S^1. Since $x \notin \mathcal{S}$ the intersection of $\mathfrak{t}^2 \oplus \{0\} \subset \mathfrak{t}^2 \oplus \mathfrak{su}(2)$ with $\mathfrak{g}_x$ is trivial. This means that $\mathfrak{g}/\mathfrak{g}_x$ splits into $\mathfrak{t}^2$, on which G_x acts trivially, and a 2-dimensional subspace $\mathfrak{m}$. As before, the normal space splits into $N_1 \oplus N_2$, where N_1 is spanned by the cross product on $\mathfrak{t}^2$ and N_2 is its orthogonal complement in N. So G_x acts trivially on N_1. To summarise, the action of G_x on T_xM splits as

$$T_xM = \mathfrak{t}^2 \oplus \mathfrak{m} \oplus N_1 \oplus N_2.$$

The action of G_x is isometric and faithful on the 2-dimensional space N_2. So, G_x is either isomorphic to $\mathrm{SO}(2)$ or to $\mathrm{O}(2)$. In the latter case, there is an element τ of order two and a subspace $N_3 \subset N_2$ that is fixed by τ. The cross products of $\mathfrak{t}^2 \oplus N_1 \oplus N_3$ generate all of T_xM so that τ acts trivially on all of T_xM. This is impossible since the action on N must be faithful.

When $\dim(G_x) = 2$, we first assume by contradiction that $x \notin \mathcal{S}$. Consider the Lie algebra homomorphism $\psi\colon \mathfrak{g}_x \to \mathfrak{su}(2)$ coming from the projection $\mathfrak{t}^2 \oplus \mathfrak{su}(2) \to \mathfrak{su}(2)$. The image of ψ would be a 2-dimensional Lie subalgebra of $\mathfrak{su}(2)$ which does not exist. It follows that $x \in \mathcal{S}$ and the identity component of G_x is isomorphic to $\mathbb{T}^2$. Since the action of the identity component of G_x splits at $T \oplus N$, we can apply Lemma 5.1.6 to see that T is isomorphic to V plus one of the W_i, for convenience say W_1, and N to the sum of $W_2 \oplus W_3$ and the statement follows.

We now deal with the $\dim(G_x) = 3$ case. Consider the Lie algebra homomorphism $\psi\colon \mathfrak{g}_x \to \mathfrak{su}(2)$ as above. The image of ψ is a Lie subalgebra of $\mathfrak{su}(2)$, hence, it is either $\mathfrak{su}(2)$ or a 1-dimensional subalgebra. The second case is impossible, indeed, the condition implies $\mathfrak{t}^2 \oplus \{0\} \subset \mathfrak{g}_x$, but $\mathfrak{g}_x$ also intersects $\mathfrak{su}(2)$ in a 1-dimensional subspace, so $\mathfrak{g}_x \cong \mathfrak{t}^2 \oplus \psi(\mathfrak{g}_x) \cong \mathfrak{t}^3$. This is a contradiction since $\mathfrak{g}_x$ is a subalgebra of $\mathfrak{g}_2$, which has rank two. So ψ is surjective, which means that $\mathfrak{g}_x$ intersects $\mathfrak{t}^2 \oplus \{0\}$ transversally. It remains to show that G_x is isomorphic to $\mathrm{SU}(2)$, which also implies that $x \notin \mathcal{S}$. As before, G_x acts trivially on $\mathfrak{g}/\mathfrak{g}_x = \mathfrak{t}^2$. The element $U_1 \times U_2$ lies in N and spans a 1-dimensional subspace N_1 on which G_x acts trivially too. On the orthogonal complement N_2 of N_1 in N the action of G_x is faithful. So G_x acts trivially on an associative three-plane, which means G_x is a subgroup of $\mathrm{SU}(2)$. Since G_x is 3-dimensional, it is isomorphic to $\mathrm{SU}(2)$ and the action on N_2 is isomorphic to the standard action of $\mathrm{SU}(2)$ on $\mathbb{C}^2$.

Finally, we consider $\dim(G_x) = 4$. Similarly as above, we can show that T is the span of U_1 and U_2, it is 1-dimensional, and it is fixed by G_x. The subgroup of G_2 that fixes a 1-dimensional subspace is $\mathrm{SU}(3)$. So, the action of G_x on the 6-dimensional normal space N defines an embedding $G_x \to \mathrm{SU}(3)$, yielding a special unitary representation of G_x on $\mathbb{C}^3$. We first show that, when restricted to the identity component, this representation must be reducible. Indeed, every 4-dimensional Lie subalgebra of $\mathfrak{g}$ is isomorphic to $\mathfrak{u}(2) = \mathfrak{su}(2) \oplus \mathfrak{u}(1)$. Since G_x is compact, it suffices to show that every complex 3-dimensional special unitary representation of $\mathrm{SU}(2) \times \mathrm{U}(1)$ is reducible. To see this, denote by V_k the unique k-dimensional irreducible representation of $\mathrm{SU}(2)$ and by W_m the representation of $\mathrm{U}(1)$ on $\mathbb{C}$ with weight m. All irreducible representations of the direct product $\mathrm{SU}(2) \times \mathrm{U}(1)$ are of the form $V_k \otimes W_m$. The 3-dimensional of these, $V_3 \otimes W_m$, are not special unitary. Since the representation is faithful and special unitary, we conclude that it must be $(V_2 \otimes W_1) \oplus W_{-2}$, i.e. of the desired form. Moreover, the element $(-1, -1)$ acts trivially, so the identity component of G_x must be $(\mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_2 \cong \mathrm{U}(2)$. $\square$

Note that the third part of the theorem implies that either $\mathcal{S}_3 = \emptyset$ or $G^{\mathrm{SU}(2)} = \mathrm{SU}(2)$.

Corollary 5.1.8. *The singular set of the* $\mathbb{T}^2$*-action is* $\mathcal{S} = \mathcal{S}_2 \cup \mathcal{S}_4$.

The following statement follows from the slice theorem and by how G_x acts on the normal bundles in Theorem 5.1.7.

Proposition 5.1.9. *Each* $\mathcal{S}_i$ *is either empty or a smooth embedded submanifold of dimension:*

$$\dim(\mathcal{S}_1) = 5, \ \dim(\mathcal{S}_2) = 3, \ \dim(\mathcal{S}_3) = 3, \ \dim(\mathcal{S}_4) = 1.$$

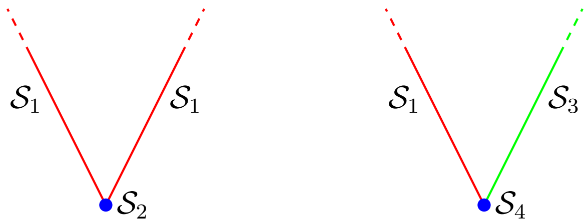


Figure 5.1: Representation of how the different $\mathcal{S}_i$s intersect.

Moreover, each connected component of $\mathcal{S}_2$ *and* $\mathcal{S}_4$ *are* G*-orbits.*

Remark 5.1.10. Note that the stratification induced by $\{\mathcal{S}_i\}$ is not the one of the orbit type stratification theorem, as there could be different orbit types of the same dimension. However, we have seen in Proposition 5.1.9 that the tangent space of each $\mathcal{S}_i$ is spanned by the tangent space of the orbit and possibly $U_1 \times U_2$. Since the flow of $U_1 \times U_2$ preserves the orbit type (see Lemma 5.3.2), the orbit type is unchanged along every connected component of each $\mathcal{S}_i$ and, hence, we can reconstruct one stratification from the other.

5.1.3 Multi-moment maps

From now on, we assume the G_2 manifold to be simply connected, so that all closed 1-forms are exact. In this setting, we can describe the components of the multi-moment maps related to φ and $*\varphi$ in an explicit way.

Remark 5.1.11. Observe that it makes sense to consider the multi-moment maps with respect to $*\varphi$ as well. Indeed, Eq. (2.2.1) implies that an action preserving φ will also preserve the metric g_φ and the volume form vol_φ. Therefore, $*\varphi$ will also be preserved.

Let U_1, U_2 be the generators of the $\mathfrak{t}^2$ component, while V_1, V_2, V_3 are the generators of the $\mathfrak{su}(2)$ component. Clearly, we can choose them to satisfy:

$$[U_l, U_m] = 0, \qquad [U_l, V_i] = 0, \qquad [V_i, V_j] = \epsilon_{ijk} V_k, \tag{5.1.2}$$

for all $l, m = 1, 2$ and $i, j, k = 1, 2, 3$.

The components of the multi-moment maps with respect to φ are defined by:

$$d\theta_i^l := \varphi(U_l, V_i, \cdot), \qquad d\nu := \varphi(U_1, U_2, \cdot), \tag{5.1.3}$$

where $l = 1, 2$, $i = 1, 2, 3$.

The components of the multi-moment maps with respect to $*\varphi$ are defined by:

$$d\mu_i := *\varphi(U_1, U_2, V_i, \cdot), \qquad d\eta := *\varphi(V_1, V_2, V_3, \cdot), \tag{5.1.4}$$

where $i = 1, 2, 3$.

As a reality check, one can show that the one-forms given on the right-hand-side are all closed.

Lemma 5.1.12. *The multi-moment maps μ and θ can be computed explicitly and, up to additive constants, have the form:*

$$\mu_k = - * \varphi(U_1, U_2, V_i, V_j), \qquad \theta^l_k = -\varphi(U_l, V_i, V_j) \tag{5.1.5}$$

i.e., $d\mu_k = *\varphi(U_1, U_2, V_k, \cdot)$ and $d\theta^l_k = \varphi(U_l, V_k, \cdot)$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$.

Proof. The proof is a straightforward application of Cartan's formula, the identity $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$ for every vector field X, Y and Eq. (5.1.2). □

Before considering the properties of the multi-moment maps, we state two classical result that we will use throughout the paper.

Lemma 5.1.13. *Let M be a smooth manifold with an $\mathrm{SU}(2)$ action of generators V_1, V_2, V_3 satisfying $[V_i, V_j] = \epsilon_{ijk} V_k$. Then, a smooth function $f\colon M \to \mathbb{R}^3$ is equivariant with respect to the action of $\mathrm{SU}(2)$ on $\mathbb{R}^3$ via the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$ if and only if f satisfies:*

$$\mathcal{L}_{V_i} f_j = \epsilon_{ijk} f_k.$$

Lemma 5.1.14. *Let M be a smooth manifold with the action of a connected Lie group G of generators $U_1, ..., U_l$. Then, a smooth function $f\colon M \to \mathbb{R}$ is invariant under the G-action if and only if f satisfies:*

$$\mathcal{L}_{U_i} f = 0,$$

for every $i = 1, ..., l$.

Proposition 5.1.15. *Let ν, $\theta^l := (\theta^l_1, \theta^l_2, \theta^l_3)$ be as in Eq. (5.1.3), let $\mu := (\mu_1, \mu_2, \mu_3)$ and η be as in Eq. (5.1.4). Then, ν is $\mathbb{T}^2 \times \mathrm{SU}(2)$-invariant, μ and θ^l are $\mathbb{T}^2$-invariant and $\mathrm{SU}(2)$-equivariant, where $\mathrm{SU}(2)$ acts on $\mathbb{R}^3$ via the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Finally, η is always $\mathrm{SU}(2)$-invariant and also $\mathbb{T}^2$-invariant if the $\mathrm{SU}(2)/\Gamma_2$-action has a singular orbit. Moreover, these functions pass to the appropriate quotients.*

Proof. The $\mathbb{T}^2$-invariance of ν, μ is clear from Lemma 5.1.14 equations Eq. (5.1.3) and Eq. (5.1.4), while the SU(2)-equivariance of μ and θ^l follows from Lemma 5.1.13 and:

$$\mathcal{L}_{V_i}\mu_j = \epsilon_{ijk}\mu_k, \qquad \mathcal{L}_{V_i}\theta^l_j = \epsilon_{ijk}\theta^l_k.$$

If we show that $\varphi(U_1, U_2, V_i) = 0$ for every $i = 1, 2, 3$, then, ν is SU(2)-invariant and θ^l is $\mathbb{T}^2$-invariant. Cartan's formula, together with $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$, implies that $d(\varphi(U_1, U_2, V_i)) = 0$ and, hence, $\varphi(U_1, U_2, V_i)$ is a constant c_i. We conclude because:

$$0 = \mathcal{L}_{V_j}c_i = V_j(\varphi(U_1, U_2, V_i)) = -\varphi(U_1, U_2, V_k) = -c_k, \tag{5.1.6}$$

where we used again Cartan's formula and Eq. (5.1.2). Analogously, one can prove that η is $\mathbb{T}^2$-invariant if the $\mathrm{SU}(2)/\Gamma_2$-action has a singular orbit. We conclude as η is obviously SU(2)-invariant. □

Since the $\mathbb{T}^2 \times \mathrm{SU}(2)$-action is structure preserving, and in particular, its generators are Killing vector fields, we can obtain the following result. Recall that the Lie derivative of a Killing vector field commutes with musical isomorphisms.

Corollary 5.1.16. *Let ν be as in Eq.* (5.1.3)*, let $\mu := (\mu_1, \mu_2, \mu_3)$ and η be as in Eq.* (5.1.4)*. Then, $\nabla\nu = U_1 \times U_2$ and $\nabla|\mu|^2$ are $\mathbb{T}^2 \times \mathrm{SU}(2)$-invariant, while $\nabla\eta$ is always* SU(2)*-invariant and also $\mathbb{T}^2$-invariant if the $\mathrm{SU}(2)/\Gamma_2$-action has a singular orbit. Moreover, these vector fields pass to the appropriate quotients.*

Remark 5.1.17. As an abuse of notation, we will use the same symbol for both the invariant functions (or vector fields) in the total space and in the quotients.

We are also able to locate the zero set of the multi-moment map of μ in terms of the stratification given in Theorem 5.1.7.

Corollary 5.1.18. *The zero set of μ satisfies:*

$$\mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4 \subset \mu^{-1}(0) \subset \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4.$$

Proof. The statement follows from Theorem 5.1.7 and and that the two-form $*\varphi(U_1, U_2, \cdot, \cdot)$ does not vanish on any 3-dimensional subspace, orthogonal to $U_1 \times U_2$. □

5.2 Local characterization of G_2 manifolds with $\mathbb{T}^2 \times \mathrm{SU}(2)$-symmetry

Any smooth hypersurface in a torsion-free G_2 manifold carries a half-flat SU(3)-structure [22]. Moreover, If this hypersurface and its structure are real-analytic one recovers the G_2-structure locally through Hitchin's flow [41]. In our setup, it is natural to take the level sets of ν as hypersurfaces, indeed, they inherit the $\mathbb{T}^2 \times \mathrm{SU}(2)$-symmetry and have $U_1 \times U_2$ as a normal vector field. The main result of this subsection, Theorem 5.2.9, is to describe half-flat SU(3)-structures with cohomogeneity one $\mathbb{T}^2 \times \mathrm{SU}(2)$-symmetry as a solution of an ODE system.

We proceed in two steps. Firstly, in Section 5.2.1, we only assume $\mathbb{T}^2$-symmetry and recall from [62] that the SU(3)-structure on the level sets of ν is described as a $\mathbb{T}^2$-bundle over a four manifold χ, with a coherent tri-symplectic structure. Secondly, in Section 5.2.2, we enhance the symmetry to $\mathbb{T}^2 \times \mathrm{SU}(2)$ which implies that the structure on χ admits a structure-preserving $G^{\mathrm{SU}(2)}$-action. In Proposition 5.2.6, we show that coherent tri-symplectic structures with this symmetry are the solution of an ODE system.

Remark 5.2.1. It is worth noting that Apostolov and Salamon [5] considered G_2 manifolds with only $\mathbb{T}^1$-symmetry. Under this weaker assumption, they still reduced their problem to a 4-manifold with an appropriate structure. In this way, they managed to construct explicit non-complete examples of G_2 manifolds.

5.2.1 $\mathbb{T}^2$-reduction

Let (M, φ) be a G_2 manifold with a $\mathbb{T}^2$ structure-preserving action and singular set $\mathcal{S}$. Associated to this action we have a multi-moment map ν, defined in Eq. (5.1.3). On $M \setminus \mathcal{S}$, the level sets of ν are hypersurfaces oriented by $\nabla\nu = U_1 \times U_2$. The $\mathbb{T}^2$-action passes to the level sets of ν and, hence, it endows $\nu^{-1}(t)$ with a $\mathbb{T}^2$-bundle structure over $\nu^{-1}(t)/\,\mathbb{T}^2$, which inherits the following additional structure (cfr. [62]).

Definition 5.2.2. A 4-manifold χ has a coherent tri-symplectic structure if it admits three symplectic forms $\overline{\sigma}_0, \overline{\sigma}_1, \overline{\sigma}_2$ such that $\overline{\sigma}_0 \wedge \overline{\sigma}_i = 0$ for $i = 1, 2$, $\overline{\sigma}_0 \wedge \overline{\sigma}_0$ is a volume form of χ and the matrix $Q := (Q_{ij})_{i,j=1,2}$ defined by $\overline{\sigma}_i \wedge \overline{\sigma}_j = Q_{ij}\overline{\sigma}_0 \wedge \overline{\sigma}_0$ is positive definite.

The forms defining this structure on $\nu^{-1}(t)/\,\mathbb{T}^2$ are:

$$\overline{\sigma}_0 = *\varphi(U_1, U_2, \cdot, \cdot), \quad \overline{\sigma}_1 = \varphi(U_1, \cdot, \cdot), \quad \overline{\sigma}_2 = \varphi(U_2, \cdot, \cdot), \tag{5.2.1}$$

where U_1, U_2 are two generators of the $\mathbb{T}^2$-action.

Conversely (see [62, Theorem 6.10]), assuming real analyticity, one can locally reconstruct a G_2 manifold with $\mathbb{T}^2$-symmetry from a coherent tri-symplectic four manifold χ, equipped with a closed two form $F \in \Omega^2(\chi, \mathbb{R}^2)$ with integral periods and whose self-dual part F_+ satisfies the orthogonality condition:

$$F_+ = (\bar{\sigma}_1, \bar{\sigma}_2)A, \tag{5.2.2}$$

for some $A \in \mathrm{GL}(2, \mathbb{R})$ such that $\mathrm{Tr}(AQ) = 0$. These conditions guarantee that F_+ is the curvature form of a $\mathbb{T}^2$-bundle N over χ. The G_2-structure is then constructed from N by running rescaled Hitchin's flow. The resulting G_2-structure yields a moment map ν of which N is a level set and rescaled Hitchin's flow evolves N into other level sets of ν.

When the symmetry is enhanced to $\mathbb{T}^2 \times \mathrm{SU}(2)$, the remaining $G^{\mathrm{SU}(2)}$-symmetry passes to the quotient χ and preserves its coherent tri-symplectic structure (see Eq. (5.2.1)). We describe such four manifolds with a free $G^{\mathrm{SU}(2)}$-symmetry, as this gives a local description in the principal part of G_2 manifolds with $\mathbb{T}^2 \times \mathrm{SU}(2)$-symmetry in terms of an explicit differential equation.

5.2.2 On 4-manifolds with a coherent symplectic triple and $G^{\mathrm{SU}(2)}$-symmetry

Let χ be a coherent symplectic 4-manifold with a $G^{\mathrm{SU}(2)}$ structure-preserving free action generated by the vector fields V_1, V_2, V_3 satisfying $[V_i, V_j] = \epsilon_{ijk} V_k$. Since the action is structure-preserving, we have that $\mathcal{L}_{V_i} \bar{\sigma}_j = 0$, therefore, Q is $G^{\mathrm{SU}(2)}$-invariant. Moreover, as Q is also positive definite, there exists a unique real symmetric, positive definite 2×2 matrix T such that $T^{-2} = T^{-1}(T^{-1})^T = Q$, which is $G^{\mathrm{SU}(2)}$-invariant as well.

Let $\mathrm{vol}_\chi := \frac{1}{2}\bar{\sigma}_0 \wedge \bar{\sigma}_0$ and define the forms $\sigma_i := \sum_{j=1}^2 T_{ij}\bar{\sigma}_j$ for $i = 1, 2$, which then satisfy $\sigma_i \wedge \sigma_j = 2\delta_{ij}\,\mathrm{vol}_\chi$. Define the metric:

$$g_\chi(u, v)\,\mathrm{vol}_\chi = \sigma_0 \wedge i_u\sigma_1 \wedge i_v\sigma_2,$$

for all $u, v \in T_x\chi$ and all $x \in \chi$. With respect to this metric, the vector fields V_i are Killing for g_χ.

Lemma 5.2.3. *There are unique g_χ-orthonormal one-forms α_i for $i = 0, ..., 3$ such that*

$$\begin{aligned} \sigma_0 &= \alpha_0 \wedge \alpha_1 + \alpha_2 \wedge \alpha_3, \quad & \sigma_1 &= \alpha_0 \wedge \alpha_2 + \alpha_3 \wedge \alpha_1, \\ \sigma_2 &= \alpha_0 \wedge \alpha_3 + \alpha_1 \wedge \alpha_2, \quad & \alpha_0 &= \frac{1}{\sqrt{\det \hat{g}_\chi}}\mathrm{vol}_\chi(V_1, V_2, V_3, \cdot), \end{aligned} \tag{5.2.3}$$

where $\hat{g}_\chi$ is the matrix $(g_\chi(V_i, V_j))_{i,j=1,2,3}$.

Proof. It suffices to prove this statement in a point. Indeed, this reduces the structure group of the bundle of coframes satisfying Eq. (5.2.3) to the trivial group. Hence, it admits a global section.

For the pointwise statement, fix a volume form on $V \cong \mathbb{R}^4$ and consider the map

$$\Lambda^2 \otimes \Lambda^2 \to \Lambda^4 \cong \mathbb{R}, \quad \alpha \otimes \beta \mapsto \alpha \wedge \beta.$$

This defines an inner product on Λ^2 with signature $(3,3)$, which gives rise to a cover $\mathrm{SL}(4,\mathbb{R}) \to \mathrm{SO}(3,3)$. The two-forms σ_is are orthogonal to each other and span a positive subspace. The statement for the first three equations of Eq. (5.2.3) follows because $\mathrm{SO}(3,3)$, and hence $\mathrm{SL}(4,\mathbb{R})$, act transitively on positive subspaces.

The stabilizer group inside $\mathrm{SL}(4,\mathbb{R})$ of the two forms in Eq. (5.2.3) is $\mathrm{SU}(2) \subset \mathrm{SO}(4)$, which acts freely and transitively on the unit sphere in $\mathbb{R}^4$. Because $\frac{1}{\sqrt{\det g_\chi}}\mathrm{vol}_\chi(V_1, V_2, V_3, \cdot)$ has unit norm, one uses the $\mathrm{SU}(2)$-action to make this one form equal to α_0. □

We define the unit vector field $X := \alpha_0^\sharp$, which satisfies the conditions $\alpha_0(X) = 1$ and $\alpha_i(X) = 0$ for $i = 1,2,3$, and determines the α_is by $\alpha_i = \sigma_{i-1}(X, \cdot)$. Consider the two 3×3-matrices:

$$\eta_{ij} := \sigma_{i-1}(X, V_j) = \alpha_i(V_j), \quad \tau_{ij} := \sigma_{i-1}(V_k, V_l),$$

where (j,k,l) is a positive permutation of $(1,2,3)$. We also define the one-forms δ_0 and δ_i for $i = 1,2,3$ by:

$$\delta_0 = \sqrt{\det \hat{g}_\chi}\alpha_0 = \mathrm{vol}_\chi(V_1, V_2, V_3, \cdot), \quad \delta_i(V_j) = \delta_{ij}, \quad \delta_i(X) = 0.$$

which satisfies $\alpha_i = \sum_{j=1}^3 \eta_{ij}\delta_j$.

Lemma 5.2.4. *The matrix functions η and τ have the following properties*

- $\tau = \mathrm{adj}(\eta^T)$,
- *the row vectors of τ and η are $G^{\mathrm{SU}(2)}$-equivariant, and hence, their determinant is $G^{\mathrm{SU}(2)}$-invariant,*
- *The metric on the vector fields V_1, V_2, V_3, which we called $\hat{g}_\chi$, is determined by η via:*

$$\hat{g}_\chi = \eta^T \eta, \tag{5.2.4}$$

- *We have the matrix equation:*

$$\sigma = \frac{1}{\det(\eta)}\delta_0 \wedge \eta\delta + \mathrm{adj}(\eta)^T \bar{\delta}, \tag{5.2.5}$$

where $\sigma = (\sigma_1, \sigma_2, \sigma_3)^T$, $\delta = (\delta_1, \delta_2, \delta_3)^T$, $\bar{\alpha} = (\alpha_2 \wedge \alpha_3, \alpha_3 \wedge \alpha_1, \alpha_1 \wedge \alpha_2)^T$ *and* $\bar{\delta} = (\delta_2 \wedge \delta_3, \delta_3 \wedge \delta_1, \delta_1 \wedge \delta_2)^T$

Proof. For the first statement, we can compute, using Eq. (5.2.3):

$$\tau_{ij} = \sigma_{i-1}(V_k, V_l) = \alpha_m \wedge \alpha_n(V_k, V_l) = \eta_{mk}\eta_{nl} - \eta_{ml}\eta_{nk} = \mathrm{adj}(\eta)_{ji},$$

where (i, m, n) is a positive permutation of $(1, 2, 3)$.

Since the vector field X commutes with V_i:

$$[V_i, X] = \mathcal{L}_{V_i}(\alpha_0^\sharp) = (\mathcal{L}_{V_i}\alpha_0)^\sharp = \frac{1}{\sqrt{\det \hat{g}_\chi}}(\mathcal{L}_{V_i}(\mathrm{vol}_\chi(V_1, V_2, V_3, \cdot))^\sharp = 0,$$

we can obtain the second statement as follows:

$$\mathcal{L}_{V_k}\eta_{ij} = \sigma_{i-1}(X, [V_k, V_j]) = -\sigma_{i-1}(X, V_l)\epsilon_{kjl} = -\eta_{il}\epsilon_{kjl}.$$

The proof is analogous for τ.

The third statement follows from the following decomposition:

$$(\hat{g}_\chi)_{ij} = g_\chi(V_i, V_j) = \sum_{k=1}^{3} \alpha_k(V_i)\alpha_k(V_j) = \sum_{k=1}^{3} \eta_{ki}\eta_{kj} = (\eta^T\eta)_{ij}.$$

For the fourth statement, observe that the equation $\alpha_i = \sum_{j=1}^{3} \eta_{ij}\delta_j$ implies that $\bar{\alpha} = \mathrm{adj}(\eta)^T\bar{\delta}$. Furthermore, Eq. (5.2.4) implies $\det(\hat{g}_\chi) = \det(\eta)^2$. In particular, η is invertible and the sign of $\det(\eta)$ does not change on χ. By swapping σ_1 and σ_2 if necessary, we can assume that $\det \eta > 0$. The formula follows from plugging these expressions into Eq. (5.2.3). $\square$

5.2.3 The differential equation

Now, we deduce how the equations $d\bar{\sigma}_i = 0$ transform under the given change of frame. We assume that $H^1(\chi, \mathbb{R}) = 0$ so that there is a function R such that $\mathrm{d}R = \delta_0$. The dual vector field ∂_R is equal to $(\det \eta)^{-1}X$, so it satisfies $[\partial_R, V_i] = 0$, for every $i = 1, 2, 3$. Morever, by Lemma 5.2.4 and the commutator relationships for X and V_i, we deduce that $\mathrm{d}\delta = -\bar{\delta}$ and $d(\frac{1}{\det \eta}\delta_0) = 0$.

We recall the following version of Lemma 5.1.13 in terms of differential forms, which can be proven using Cartan's formula.

Lemma 5.2.5. *A smooth function* $f\colon \chi \to \mathbb{R}^3$ *is* $\mathrm{SU}(2)$*-equivariant if and only if* $(df = f \times \delta) \mod \delta_0$, *for* $(f \times \delta)_i = \epsilon_{ijk} f_j \delta_k$.

Consequently, we have

$$d\eta = \eta \times \delta + \frac{\partial \eta}{\partial R}\delta_0, \quad d\tau = \tau \times \delta + \frac{\partial \tau}{\partial R}\delta_0,$$

where $(\eta \times \delta)_{ij} = (\eta_i \times \delta)_j$ and $(\tau \times \delta)_{ij} = (\tau_i \times \delta)_j$, i.e. we are taking the cross products of the rows of η with δ. Putting all together in Eq. (5.2.5), we get

$$d\sigma = \frac{1}{\det \eta}\delta_0 \wedge (-d\eta \wedge \delta - \eta d\delta) + d\tau \wedge \bar{\delta} = \frac{1}{\det \eta}\delta_0 \wedge (-\eta\bar{\delta}) + (\partial_R \tau)\delta_0 \wedge \bar{\delta}.$$

The last step is due to the two identities:

$$(\eta \times \delta) \wedge \delta = 2\eta\bar{\delta}, \quad (\tau \times \delta) \wedge \bar{\delta} = 0.$$

Extend T to a 3×3 matrix by padding it with one in the $(1,1)$ entry and by zeros in the first row and column elsewhere. This extension is such that $\sigma = T\bar{\sigma}$, which implies:

$$\mathrm{d}\sigma = \mathrm{d}T \wedge \bar{\sigma} = \partial_R(T)T^{-1}\delta_0 \wedge \sigma = \partial_R(T)T^{-1}\tau\delta_0 \wedge \bar{\delta}. \tag{5.2.6}$$

Combining the two equations for $d\sigma$ and using $\frac{1}{\det \eta}\eta = (\tau^T)^{-1}$ gives:

$$0 = (\partial_R \tau - (\partial_R T)T^{-1}\tau - (\tau^T)^{-1})\delta_0 \wedge \bar{\delta}. \tag{5.2.7}$$

Proposition 5.2.6. *A coherent symplectic 4-manifold χ with free $G^{\mathrm{SU}(2)}$-symmetry and intersection matrix Q admits a matrix-valued function $\tau\colon \chi \to M_{3\times 3}(\mathbb{R})$ whose rows are equivariant with respect to the action of* SO(3) *on $\mathbb{R}^3$ and satisfying the following differential equation:*

$$\partial_R \tau = (\partial_R T)T^{-1}\tau + (\tau^T)^{-1}, \tag{5.2.8}$$

where $T : \chi \to M_{3\times 3}(\mathbb{R})$ is the, padded as above, matrix satisfying $Q = T^{-2}$.

Conversely, given a function $T : (a,b) \to \mathrm{Sym}_{2\times 2}(\mathbb{R})$ of positive-definite matrices, identified with $T : (a,b) \to \mathrm{Sym}_{3\times 3}(\mathbb{R})$ padded as above, then, every equivariant solution $\tau : (a,b) \times G^{\mathrm{SU}(2)} \to M_{3\times 3}(\mathbb{R})$ of Eq. (5.2.8) *defines a coherent symplectic structure on $(a,b) \times G^{\mathrm{SU}(2)}$ with intersection matrix $Q = T^{-2}$.*

Proof. The first statement follows from Eq. (5.2.7) since the $\delta_0 \wedge \bar{\delta}_i$ are linearly independent on χ.

For the converse direction, define the frame $\delta_0, \ldots, \delta_3$ on $(a,b) \times \mathrm{SU}(2)$ such that $\delta_0 = dR$ and δ_i are the invariant one-forms on SU(2), hence, satisfying $d\delta_i = -\epsilon_{ijk}\delta_j \wedge \delta_k$. Lemma 5.2.5 and Eq. (5.2.8) imply

$$\mathrm{d}\tau = \tau \times \delta + \left((\partial_R T)T^{-1}\tau + (\tau^T)^{-1})\right)\delta_0 \tag{5.2.9}$$

Define the forms α_i by the equation $\alpha_i = \sum_{j=1}^{3} \eta_{ij}\delta_j$, with $\eta := \mathrm{adj}(\tau^T))$ as before. From the α_is, we can reconstruct the forms σ by Eq. (5.2.3) and then $\bar{\sigma}$ through the transformation matrix T. We deduce that $\bar{\sigma}_i$ are such that $\bar{\sigma}_0 \wedge \bar{\sigma}_i = 0$ and $\bar{\sigma}_i \wedge \bar{\sigma}_j = Q_{ij}\frac{1}{2}\sigma_0 \wedge \sigma_0$, where $Q = T^{-2}$. Our previous computations show that Eq. (5.2.9) implies that the forms $\bar{\sigma}_i$ are closed and, hence, we conclude. □

Remark 5.2.7. If Q is the identity matrix, then g_χ is hyperkähler and by rotating $\sigma_0, \sigma_1, \sigma_2$ we can assume that τ is a diagonal at a given point. The diagonality is preserved along R (as in the Biachi IX ansatz) by Eq. (5.2.8), and we have $\partial_R \frac{1}{2}\tau_{ii}^2 = 1$ for $i = 1, 2, 3$. So each τ_{ii} is of the form $\sqrt{2R + k_i}$ and can we assume that $k_1 + k_2 + k_3 = 0$ and $k_1 \geq k_2 \geq k_3$. The metric g_χ is

$$\frac{1}{\tau_{11}\tau_{22}\tau_{33}}\mathrm{d}R^2 + \frac{\tau_{22}\tau_{33}}{\tau_{11}}\delta_1^2 + \frac{\tau_{33}\tau_{11}}{\tau_{22}}\delta_2^2 + \frac{\tau_{11}\tau_{22}}{\tau_{33}}\delta_3^2$$

If all $k_i = 0$, then, all τ_{ii} are equal and the metric is flat. If $k_1 > 0$ and $k_2 = k_3 < 0$ then g_χ is the Eguchi-Hanson metric. In all other cases the metric is incomplete. Note that the Taub-NUT and Atiyah-Hitchin metric are not described by our set-up, since the SU(2) action is not tri-holomorphic on these spaces. Instead, the action rotates the three hyperkähler two-forms.

We refer the reader to [7] for further details on hyperkähler metrics in 4-dimensions.

5.2.4 From coherent tri-symplectic manifolds to G_2 manifolds

Finally, we use Proposition 5.2.6 to obtain a local construction of G_2 manifolds with $\mathbb{T}^2 \times \mathrm{SU}(2)$-symmetry through [62, Theorem 6.10]. The last object that we need is an orthogonal self-dual two-form $F_+ \in \Omega^2(\chi, \mathbb{R}^2)$ on χ with integral periods. This condition assumes the existence of an anti-self-dual form $F_- \in \Omega^2(\chi, \mathbb{R}^2)$ such that $F_+ + F_-$ is closed and defines an element in $H^2(M, \mathbb{Z}^2)$. In the $G^{\mathrm{SU}(2)}$-invariant case the closedness condition can always be satisfied.

Lemma 5.2.8. *For any $G^{\mathrm{SU}(2)}$-invariant $F_+ \in \Omega^2_+(\chi, \mathbb{R}^2)$, there is a $F_- \in \Omega^2_-(\chi, \mathbb{R}^2)$ such that $F_+ + F_-$ is closed.*

Proof. Define the anti-self dual two forms that are analogous to the σ_is we defined above:

$$\begin{aligned}
\sigma_0^- &= -\alpha_0 \wedge \alpha_1 + \alpha_2 \wedge \alpha_3, \\
\sigma_1^- &= -\alpha_0 \wedge \alpha_2 + \alpha_3 \wedge \alpha_1, \\
\sigma_2^- &= -\alpha_0 \wedge \alpha_3 + \alpha_1 \wedge \alpha_2.
\end{aligned}$$

Then σ^- satisfies the same structure equation Eq. (5.2.6). This can be shown by computing $\mathrm{d}\sigma^-$ as before or by using a local diffeomorphism that preserves $\alpha_1, \alpha_2, \alpha_3$ and flips the sign of α_0, i.e pulls back σ to σ^-. This implies:

$$\mathrm{d}(\sigma - \sigma^-) = \partial_R(T)T^{-1}\delta_0 \wedge (\sigma - \sigma^-),$$

which vanishes as $\sigma - \sigma^- = 2\alpha_0 \wedge \alpha$ and α_0 is proportional to δ_0. Since F_+ is self-dual, there is $a\colon \chi \to \mathbb{R}^3$ such that $F_+ = a\sigma = \sum a_i\sigma_i$. Because F_+ is $G^{\mathrm{SU}(2)}$-invariant, the same is true for a, which means $\mathrm{d}a$ is a multiple of α_0. Now define $F_- := -a\sigma^-$ and observe

$$\mathrm{d}(F_+ + F_-) = \sum_{i=1}^{3} 2\mathrm{d}a_i \wedge \alpha_0 \wedge \alpha_i = 0,$$

as required. $\square$

If the function T is real-analytic the solution to Eq. (5.2.8) is too by the Cauchy-Kovalevskaya theorem. Clearly if F_+ is real-analytic, so is F_- and also the half-flat $\mathrm{SU}(3)$-structure constructed in [62, Proposition 6.5]. This observation, together with Proposition 5.2.6 and [62, Theorem 6.10] implies the following theorem.

Theorem 5.2.9 (Aslan–T. [6]). *Let $T : (a,b) \to \mathrm{Sym}_{3\times 3}(\mathbb{R})$ as in Proposition 5.2.6, and let $F_+ \in \Omega^2_+((a,b) \times G^{\mathrm{SU}(2)}, \mathbb{R}^2)$ satisfying Eq. (5.2.2) and such that $F_+ + F_-$ from Lemma 5.2.8 has integral periods. Then, there is a torus bundle $N \to (a,b) \times G^{\mathrm{SU}(2)}$ and every equivariant solution $\tau : (a,b) \times G^{\mathrm{SU}(2)} \to M_{3\times 3}(\mathbb{R})$ of Eq. (5.2.8) defines an half-flat $\mathrm{SU}(3)$-structure on N, which admits a $\mathbb{T}^2 \times \mathrm{SU}(2)$-symmetry. Moreover, if the coefficient function T and F_+ are real-analytic, this induces a torsion-free G_2-structure on $(-\epsilon, \epsilon) \times N$ admitting the same symmetry.*

The equations can be viewed on the quotient B, parametrised by $|\mu|$ and ν. Indeed $(\mathrm{d}\nu)^\flat$ is the direction of rescaled Hitchin's flow. Furthermore, for the coherent symplectic structure on χ_t, we have

$$\begin{aligned}\mathrm{d}R = \mathrm{vol}_\chi(V_1, V_2, V_3, \cdot) &= \sum_{i,j,k} \frac{\epsilon_{ijk}}{2} * \varphi(U_1, U_2, V_i, V_j) * \varphi(U_1, U_2, V_k, \cdot) = -\sum_k \mu_k \mathrm{d}\mu_k \\ &= -\frac{1}{2}\mathrm{d}|\mu|^2.\end{aligned}$$

So, up to a constant, $R = \frac{1}{2}|\mu|^2$.

5.3 $\mathbb{T}^2$-invariant associative submanifolds

In this section, we study $\mathbb{T}^2$-invariant associative submanifolds of the G_2 manifold (M, φ), endowed with a structure-preserving, cohomogeneity two action of $\mathbb{T}^2 \times \mathrm{SU}(2)$ on it. We use the same notation and conventions of Section 5.1.

5.3.1 $\mathbb{T}^2$-invariant associatives

As in Section 5.1.3, let U_1 and U_2 be the generators of the $\mathfrak{t}^2$ component in $\mathfrak{t}^2 \times \mathfrak{su}(2)$. We give a first characterization of $\mathbb{T}^2$-invariant associatives as integral curves of a vector field.

Proposition 5.3.1. *Let L_0 be a $\mathbb{T}^2$-invariant associative submanifold of $M \setminus \mathcal{S} \subseteq M_P$. Then, $L_0/\mathbb{T}^2$ is an integral curve of the nowhere vanishing vector field $U_1 \times U_2$ in $(M \setminus \mathcal{S})/\mathbb{T}^2$. Conversely, every integral curve of $U_1 \times U_2$ in $(M \setminus \mathcal{S})/\mathbb{T}^2$ is the projection of a $\mathbb{T}^2$-invariant associative in $M \setminus \mathcal{S}$.*

Proof. Since U_1, U_2 are linearly independent in $M \setminus \mathcal{S}$, the vector field $U_1 \times U_2$ is nowhere vanishing there, we deduce that $\{U_1, U_2, U_1 \times U_2\}$ is an associative plane from Proposition 2.2.8. The statement follows immediately from the correspondence between curves in $(M \setminus \mathcal{S})/\mathbb{T}^2$ and $\mathbb{T}^2$-invariant 3-submanifolds in $M \setminus \mathcal{S}$. □

We now state some general properties of $\mathbb{T}^2$-invariant associatives and integral curves of $U_1 \times U_2$ that will play a crucial role later on. Since the flow of $U_1 \times U_2$ commutes with the group action of G, we have the following.

Lemma 5.3.2. *The flow along $U_1 \times U_2$ preserves the orbit type of G. Therefore, integral curves of $U_1 \times U_2$ stay in the same stratum of the stratification of the orbit type stratification theorem, and hence of the one described in Theorem 5.1.7.*

In particular, we have proven that the problem of finding $\mathbb{T}^2$-invariant associatives decomposes with respect to the stratification, and, on $M \setminus \mathcal{S}$ it reduces to a problem of finding integral curves of a nowhere vanishing vector field.

Lemma 5.3.3. *The multi-moment map $\mu : M \to \mathbb{R}$ is preserved by the vector field $U_1 \times U_2$. Therefore, μ is constant on every $\mathbb{T}^2$-invariant associative.*

Proof. By definition of μ_i we have $d\mu_i(U_1 \times U_2) = *\varphi(U_1, U_2, V_i, U_1 \times U_2)$ for every $i = 1, 2, 3$. If U_1, U_2 are linearly independent, then, $\{U_1, U_2, U_1 \times U_2\}$ is an associative plane and $*\varphi(U_1, U_2, V_i, U_1 \times U_2) = 0$ by Proposition 2.2.8. Otherwise, the equation trivially holds. □

5.3.2 Associatives in the principal set

In this subsection, we restrict our attention to the principal set M_P. Let U_1, U_2, V_1, V_2, V_3 be the generators of the G-action as in Section 5.1.3. Note that the action is assumed to be of cohomogeneity two, hence, the generators are everywhere linearly independent on M_P.

Proposition 5.3.4. *The restriction of (μ, ν) to M_P is a submersion. In particular, $\mu^{-1}(c) \cap M_P$ is a 4-dimensional submanifold of M_P for every c in the image $\mu(M_P)$ and $(|\mu|, \nu) : M_P/G \to \mathbb{R}^2$ is a local diffeomorphism onto its image.*

Proof. Given a fixed $x \in M_P$, it follows from Corollary 5.1.18 that $\mu(x) \neq 0$. Since μ is SU(2)-equivariant and ν is SU(2)-invariant, it suffices to show that $(|\mu|^2, \nu)$ is a submersion at x.

As $\sum_{k=1}^3 \varphi(U_1, U_2, \mu_k V_k) = 0$, there is an $X \in T_x M$ such that $\sum_{k=1}^3 *\varphi(U_1, U_2, V_k \mu_k, X) = 1$. Observe that

$$\frac{1}{2}\mathrm{d}|\mu|^2 = \sum_{k=1}^{3} \mu_k * \varphi(U_1, U_2, V_k, \cdot),$$

which implies $\mathrm{d}|\mu|^2(X) = 2$ and $\mathrm{d}|\mu|^2(U_1 \times U_2) = 0$. The statement follows because $\mathrm{d}\nu(U_1 \times U_2) \neq 0$ on M_P. □

We now take a different perspective. Indeed, we argued in Lemma 5.1.5 that the action of SU(2) on M induces on the quotient $M_P/\mathbb{T}^2$ a principal bundle structure with structure group $G^{\mathrm{SU}(2)}$ and base space the surface B. Let $\mathcal{H}$ be a connection on $M_P/\mathbb{T}^2$ such that the SU(2)-invariant $U_1 \times U_2$ is horizontal at each point. A connection satisfying this property always exists, indeed, we showed in Proposition 5.1.15 that the one induced by the G_2-metric satisfies:

$$0 = g(U_1 \times U_2, V_j) = \varphi(U_1, U_2, V_j).$$

Using such a connection, integral curves of $U_1 \times U_2$ are horizontal lifts over such curves in B.

Theorem 5.3.5 (Aslan–T. [6]). *Let $\mathcal{H}$ be a connection on the principal $G^{\mathrm{SU}(2)}$-bundle $M_P/\mathbb{T}^2 \to B$ such that $U_1 \times U_2 \in \mathcal{H}$. Let γ be a curve in $M_P/\mathbb{T}^2$. The following are equivalent:*

1. *The pre-image $\pi_{\mathbb{T}^2}^{-1}(\mathrm{im}\gamma)$ is a $\mathbb{T}^2$-invariant associative in M_P,*
2. *γ is an integral curve of $U_1 \times U_2$,*

3. γ is the horizontal lift of a level set of $|\mu|$ on B.

Moreover, the correspondence between (1) and (2) is 1-to-1, while for every integral curve of $U_1 \times U_2$ in B there is a $G^{\mathrm{SU}(2)}$-family of integral curves of $U_1 \times U_2$ in $M_P/\mathbb{T}^2$.

Proof. The equivalence between (1) and (2) has been established in Proposition 5.3.1, while the equivalence between (2) and (3) can be deduced from the G-invariance of $U_1 \times U_2$, the fact that it is assumed to be horizontal and Proposition 5.3.4. □

5.3.3 Local description of associatives in the principal set

We have seen that $M_P/\mathbb{T}^2$ is a $G^{\mathrm{SU}(2)}$-principal bundle over the base B. In Theorem 5.3.5, the integral curves of $U_1 \times U_2$ in $M_P/\mathbb{T}^2$ are described as horizontal lifts of curves in a surface. In the following, we will show how these horizontal lifts can be computed in a local trivialization of the principal bundle.

Lemma 5.3.6. *Let $\mathcal{U} \times G^{\mathrm{SU}(2)} \to M_P/\mathbb{T}^2$ be a local trivialisation with $U_1 \times U_2 \in T\mathcal{U} \times \{0\}$, inducing a local chart $\bar{\mathcal{U}} \subset M_P$ and a projection map $p_{G^{\mathrm{SU}(2)}} : \bar{\mathcal{U}} \to G^{\mathrm{SU}(2)}$. Then, the fibres of the submersion $(|\mu|, p_{G^{\mathrm{SU}(2)}}) : \bar{\mathcal{U}} \to \mathbb{R}^+ \times G^{\mathrm{SU}(2)}$ are associative submanifolds.*

Proof. As $U_1 \times U_2 \in T\mathcal{U} \times \{0\}$, it follows that its integral curves will be constant on the $G^{\mathrm{SU}(2)}$ component of $\mathcal{U} \times G^{\mathrm{SU}(2)}$. Since $|\mu|$ is constant on the $G^{\mathrm{SU}(2)}$-component and since integral curves of $U_1 \times U_2$ are contained in the level set of $|\mu|$ we conclude the proof. □

The aim is to find trivializations of $M_P/\mathbb{T}^2 \to B$ where we can apply Lemma 5.3.6. Since μ is $G^{\mathrm{SU}(2)}$-equivariant, we can reduce the structure group of the $G^{\mathrm{SU}(2)}$-principal bundle. Indeed, given $v \in \mathbb{R}^3 \setminus \{0\}$ and denoting by $\langle v \rangle$ the line spanned by v, then, $Q_v := \mu^{-1}(\langle v \rangle)$ is an S^1 reduction of the bundle $M_P/\mathbb{T}^2 \to B$.

Proposition 5.3.7. *In a neighbourhood $\mathcal{U} \subset B$, where $(|\mu|, \nu)$ is a diffeomorphism onto its image and the image is convex, there exists a flat connection on Q_v such that $U_1 \times U_2$ is horizontal.*

Proof. Let $\theta \in \Omega^1(Q_v, \mathbb{R})$ be any connection form on Q_v for which $U_1 \times U_2$ is horizontal. Then the curvature form $d\theta$ is a basic form, so there is a function $f : \mathcal{U} \to \mathbb{R}$ such that $d\theta = f d\nu \wedge d|\mu|$, where we are considering $(|\mu|, \nu)$ as coordinates on $\mathcal{U} \subset B$. The form $d|\mu|$ is basic and annihilates $U_1 \times U_2$, hence, $\theta' = \theta + F d|\mu|$ is also a connection on Q_v such that $U_1 \times U_2$ is horizontal for every smooth function $F : \mathcal{U} \to \mathbb{R}$. The new connection θ' is flat if and only if $(\partial_\nu F + f) d\nu \wedge d|\mu| = 0$. Because the image is convex, $\partial_\nu F = -f$ admits at least one solution, for instance, using the methods of characteristics. □

Theorem 5.3.8 (Aslan–T. [6]). *In a neighbourhood $\mathcal{U} \subset B$ where $(|\mu|, \nu)$ is a diffeomorphism onto its image, and the image is convex, there exists a trivialization $\mathcal{U} \times G^{\mathrm{SU}(2)} \to M_P/\mathbb{T}^2$ such that $U_1 \times U_2 \in T\mathcal{U} \times \{0\}$. As a consequence, the map $(|\mu|, p_{G^{\mathrm{SU}(2)}})$ is a fibre bundle map whose fibres are associative submanifolds. Here, $p_{G^{\mathrm{SU}(2)}}$ is the projection to $G^{\mathrm{SU}(2)}$ coming from the trivialisation.*

Proof. The bundle Q_v has a flat connection for which $U_1 \times U_2$ is horizontal. Since $\mathcal{U}$ is simply-connected, there is a trivialization $\mathcal{U} \times S^1 \to Q_v$ which induces this connection, i.e. the horizontal bundle is $T\mathcal{U} \times \{0\} \subset TQ_v$. Since $U_1 \times U_2$ is horizontal the component in S^1 is constant along integral curves of $U_1 \times U_2$. By equivariance, we get a trivialization $\mathcal{U} \times G^{\mathrm{SU}(2)} \to M_P/\mathbb{T}^2$ such that the component in $G^{\mathrm{SU}(2)}$ is constant along integral curves of $U_1 \times U_2$. We conclude using Lemma 5.3.6 and because the image of $(|\mu|, \nu)$ is convex. $\square$

Clearly, the condition on $(|\mu|, \nu)$ in Theorem 5.3.8 always holds locally.

5.3.4 Associatives in the singular set

In this subsection, we describe the $\mathbb{T}^2$-invariant associative submanifolds of M that are contained in the singular set of the $\mathbb{T}^2 \times \mathrm{SU}(2)$-action. In particular the following theorem holds.

Theorem 5.3.9 (Aslan–T. [6]). *Let $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ and $\mathcal{S}_4$ be the strata as described in Theorem 5.1.7. Then,*

- *$\mathcal{S}_1$ admits an $\mathrm{SU}(2)$-equivariant submersion $F \colon \mathcal{S}_1 \to S^2$ or $F \colon \mathcal{S}_1 \to \mathbb{RP}^2$ such that each (not necessarily connected) fibre is a $\mathbb{T}^2$-invariant totally geodesic associative.*
- *every connected component of $\mathcal{S}_2$ is an associative G-orbit,*
- *The set $\mathcal{S}_3 \cup \mathcal{S}_4$ is totally geodesic, associative and the action of G on $\mathcal{S}_3$ is of cohomogeneity one.*

Proof. We first consider $\mathcal{S}_1$. For every $\underline{c} \in \mathbb{R} \times \mathbb{R}$ and $\underline{b} \in S^2$, consider the Killing vector field $W_{\underline{c},\underline{b}} := c_1 U_1 + c_2 U_2 + b_1 V_1 + b_2 V_2 + b_3 V_3$ and its zero set $L_{\underline{c},\underline{b}} \subset M \setminus \mathcal{S}$. Observe that every point of $\mathcal{S}_1$ lies in a unique $L_{\underline{c},\underline{b}}$, up to $L_{\underline{c},\underline{b}} = L_{-\underline{c},-\underline{b}}$. Indeed, $W_{\underline{c},\underline{b}}$ corresponds to the Lie algebra of $G_x \cong S^1$. If $L_{\underline{0},\underline{b}}$ is non-empty, then we define $F : \mathcal{S}_1 \to \mathbb{RP}^2$ by mapping each point to the corresponding $[\underline{b}] \in \mathbb{RP}^2$.

If $L_{\underline{0},\underline{b}}$ is empty, then the map can be lifted to $F : \mathcal{S}_1 \to S^2$. This can be done because $\underline{c}$ fixes the $\mathbb{Z}_2$-action as follows. As G_x is the quotient of a compact 1-dimensional subgroup of $\mathbb{T}^2 \times \mathrm{SU}(2)$ and $W_{\underline{c},\underline{b}}$ spans its Lie algebra, we deduce that $\underline{c} \in \mathbb{Q} \times \mathbb{Q} \setminus \{\underline{0}\}$, (otherwise, $L_{\underline{c},\underline{b}}$ is empty). Any line in $\mathbb{R}^2$ determines two open half-spaces which are such that $-\mathrm{Id}_{\mathbb{R}^2}$ is bijective. Moreover, if the line is chosen to be of irrational slope, then it does not intersect $\mathbb{Q} \times \mathbb{Q} \setminus \{\underline{0}\}$. Let H^+ be one of these half-planes. Now, every $x \in \mathcal{S}_1$ is in $L_{\underline{c},\underline{b}} = L_{-\underline{c},-\underline{b}}$ for some $(\underline{c}, \underline{b}) \in \mathbb{Q}^2 \setminus \{\underline{0}\} \times S^2$, but only one element of $\{\pm\underline{c}\}$ is in H^+. Hence, we argued that

$$\mathcal{S}_1 = \bigcup_{(\underline{c},\underline{b}) \in H^+ \times S^2} L_{\underline{c},\underline{b}}$$

and that the union is disjoint. We define $F : \mathcal{S}_1 \to S^2$ such that on each of $L_{\underline{c},\underline{b}}$ the value of F is $\underline{b}$.

To show that F is equivariant, let $\xi_{c,b}$ be the Lie algebra element corresponding to the vector field $W_{c,b}$ and recall that

$$L_{\underline{c},\underline{b}} = \{x \in M \mid \xi_{c,b} \in \mathfrak{g}_x\},$$

where $\mathfrak{g}_x$ is the Lie algebra of G_x. The equivariance follows because, for every $g \in \mathrm{SU}(2)$ we have:

$$\xi_{c,b} \in \mathfrak{g}_x \quad \Leftrightarrow \quad \xi_{c,gb} = \mathrm{Ad}_g \xi_{c,b} \in \mathrm{Ad}_g \mathfrak{g}_x = \mathfrak{g}_{gx}$$

The space $L_{\underline{c},\underline{b}}$ is a totally geodesic submanifold since it is the zero set of a Killing vector field and, since the vector fields $U_1, U_2, U_1 \times U_2$ commute with $W_{c,b}$, they are linearly independent and tangent to $L_{\underline{c},\underline{b}}$. It remains to show that F is a submersion. For a point $x \in \mathcal{S}_1$, a neighbourhood of the orbit Gx in $\mathcal{S}_1$ is diffeomorphic to $\mathbb{R} \times G/G_x$. The vector field $U_1 \times U_2$ is tangent to the $\mathbb{R}$ direction, so F is invariant under the coordinate in $\mathbb{R}$ and descends to a G-equivariant map G/G_x, which is a $\mathbb{T}^2$-invariant submersion.

We now turn our attention to $\mathcal{S}_2$. By Proposition 5.1.9, $\mathcal{S}_2$ is smooth, 3-dimensional and, by Theorem 5.1.7, associative. As it is 3-dimensional, we deduce that every connected component is a G-orbit.

Finally, we consider $\mathcal{S}_3 \cup \mathcal{S}_4$. In Proposition 5.1.9, we have seen that $\mathcal{S}_3$ is smooth and 3-dimensional and that $\mathcal{S}_4$ is smooth and 1-dimensional. It follows that $\mathcal{S}_3$ is dense in $\mathcal{S}_3 \cup \mathcal{S}_4$ and it suffices to show that $\mathcal{S}_3 \cup \mathcal{S}_4$ is smooth and that $\mathcal{S}_3$ is associative, totally geodesic and of cohomogeneity one. Clearly, $\mathcal{S}_3$ is open in $\mathcal{S}_3 \cup \mathcal{S}_4$. Hence, it is enough to show smoothness at a point $x \in \mathcal{S}_4$. By Theorem 5.1.7, the normal representation of G_x on $\mathbb{C}^3$ splits into two invariant components $N = N_1 \oplus N_2$ where $\dim(N_1) = 1, \dim(N_2) = 2$. The set of points with 3-dimensional stabilizer is exactly N_1. So, by the slice theorem, there is a diffeomorphism of $Gx \times N$ to a neighbourhood $U \subset M$ of Gx such that $Gx \times N_1$ is mapped to $U \cap (\mathcal{S}_3 \cup \mathcal{S}_4)$ and smoothness follows.

The set $\mathcal{S}_3$ is totally geodesic because it is the common zero locus of three Killing vector fields. The submanifold $\mathcal{S}_3$ is associative because at each point the tangent space is the span of U_1, U_2 and $U_1 \times U_2$. $\square$

Combining Theorem 5.3.8 with Theorem 5.3.9 we obtain an associative fibration in the sense of Definition 3.1.5.

Corollary 5.3.10. *If $(|\mu|, \nu)\colon B \to \mathbb{R}^2$ is a diffeomorphism onto its image with fibres of ν connected, then M admits a global $\mathbb{T}^2$-invariant associative fibration.*

5.3.5 Singularity analysis

In this last subsection, we show that every $\mathbb{T}^2$-invariant associative in a G_2 manifold with $\mathbb{T}^2 \times \mathrm{SU}(2)$-symmetry needs to be smooth.

Theorem 5.3.11 (Aslan–T. [6])**.** *Every $\mathbb{T}^2$-invariant φ-calibrated current in M is a smooth submanifold. Moroever, if a $\mathbb{T}^2$-invariant φ-calibrated current intersects the singular set of the $\mathbb{T}^2 \times \mathrm{SU}(2)$-action, then, it is contained in the singular set.*

Proof. As a first step, we observe that the local uniqueness and existence theorem implies that $\mathbb{T}^2$-invariant φ-calibrated currents are smooth away from $\mathcal{S} = \mathcal{S}_2 \cup \mathcal{S}_4$.

Moreover, if L is a $\mathbb{T}^2$-invariant φ-calibrated current intersecting $\mathcal{S}$, we claim that it needs to be contained in the singular set of the G-action. Indeed, if by contradiction $\operatorname{supp} L \cap M_P \neq 0$, then, $\mu\big|_{\operatorname{supp} L} = c$ for some constant $c \neq 0$, by Corollary 5.1.18. However, once again by Corollary 5.1.18, we have that $\mu\big|_{\mathcal{S}_2 \cup \mathcal{S}_4} = 0$ which is a contradiction as μ is constant on L.

All we are left to do is to consider: $L \subset \mathcal{S}_1 \cup \mathcal{S}$ and not completely contained in $\mathcal{S}$. Note that the smoothness of $L \subset \mathcal{S}_3 \cup \mathcal{S}_2 \cup \mathcal{S}_4$ was proven in Theorem 5.3.9. Now, given $x \in \mathcal{S}_1 \cap L \neq \emptyset$ we can associate a unique vector field $W_{\underline{c},\underline{b}}$ on M, such that its zero set in $\mathcal{S}_1$ coincides with $L \cap \mathcal{S}_1$ (or one of its connected components). We conclude that L is globally the zero set of a Killing vector field $W_{\underline{c},\underline{b}}$, which is a smooth totally geodesic submanifold. $\square$

Remark 5.3.12. The approach used to study the singularities in Theorem 5.4.5 and Theorem 5.4.19 can be attempted for $\mathbb{T}^2$-invariant associatives as well. However, in this case, we could not rule out the existence of branched points.

5.4 $\mathbb{T}^3$-invariant and $\mathrm{SU}(2)$-invariant coassociative submanifolds

In this section, we study coassociative submanifolds of the G_2 manifold (M, φ), endowed with a structure-preserving, cohomogeneity two action of $\mathbb{T}^2 \times \mathrm{SU}(2)$ on it. We use the same notation and conventions of Section 5.1.1. In particular, we consider coassociative submanifolds that are invariant under $\mathbb{T}^3 = \mathbb{T}^2 \times S^1 \subset \mathbb{T}^2 \times \mathrm{SU}(2)$, for some $S^1 \subset \mathrm{SU}(2)$, and $\mathrm{SU}(2) = \mathrm{Id}_{\mathbb{T}^2} \times \mathrm{SU}(2) \subset \mathbb{T}^2 \times \mathrm{SU}(2)$.

5.4.1 $\mathbb{T}^3$-invariant coassociative submanifolds

Given any $S^1 \subset \mathrm{SU}(2)$, we can consider a structure preserving $\mathbb{T}^3$-action on M by $\mathbb{T}^2 \times S^1 \subset \mathbb{T}^2 \times \mathrm{SU}(2)$. Moreover, up to passing to some quotient, we can assume that the action is effective. We denote by $\overline{\mathcal{S}}$ the singular set of this action which satisfies: $\mathcal{S}_2 \cup \mathcal{S}_4 \subseteq \overline{\mathcal{S}} \subseteq \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$. Madsen and Swann proved in [64, Lemma 2.6] that the stabilizer of an effective $\mathbb{T}^3$-action on a G_2 manifold is either trivial, a circle or a two-torus.

In the notation of Section 5.1.3, we can assume that the generators of the $\mathbb{T}^3$ action are U_1, U_2, V_1 and, hence, the multi-moment maps associated to it are $\mu_1, \theta_1^1, \theta_1^2$ and ν. Similarly to the $\mathbb{T}^2$-invariant associative case, we can see $\mathbb{T}^3$-invariant coassociatives as integral curves of a vector field.

Proposition 5.4.1. *Let Σ_0 be a $\mathbb{T}^3$-invariant coassociative submanifold of $M \setminus \overline{\mathcal{S}}$. Then, $\Sigma_0/\mathbb{T}^3$ is an integral curve of the nowhere vanishing vector field $\nabla\mu_1$ in $(M \setminus \overline{\mathcal{S}})/\mathbb{T}^3$. Conversely, every integral curve of $\nabla\mu_1$ in $(M \setminus \overline{\mathcal{S}})/\mathbb{T}^3$ is the projection of a $\mathbb{T}^3$-invariant coassociative in $M \setminus \overline{\mathcal{S}}$.*

Differently from the associative case, $\nabla\mu_1$ does not commute with $\mathbb{T}^2 \times \mathrm{SU}(2)$, hence, integral curves do not respect the stratification of Section 5.1.2. However, the following holds.

Lemma 5.4.2. *Let γ be an integral curve of $\nabla\mu_1$ in $M \setminus \overline{\mathcal{S}}$. Then, the multi-moment map μ_1 is strictly increasing along γ.*

We recall that $\mathbb{T}^3$-invariant coassociatives are the level sets of the following multi-moment maps.

Proposition 5.4.3 (Madsen–Swann [64]). *The map $(\theta_1^1, \theta_1^2, \nu) : M \setminus \overline{\mathcal{S}} \to \mathbb{R}^3$ is a submersion with fibres $\mathbb{T}^3$-invariant coassociative submanifolds.*

Remark 5.4.4. Differently from the $\mathbb{T}^2$-invariant associative case, where we showed that M admits an associative fibration in the sense of Definition 3.1.5, we can not argue in the same way in this case. Indeed, a priori we do not know if there exists a $\mathbb{T}^3$-invariant coassociative passing through each point of $\overline{\mathcal{S}}$.

Using a completely different approach to the one employed in Theorem 5.3.11, we can study the singularities that a $\mathbb{T}^3$-invariant coassociative can develop. To this scope, we need to describe the structure of the local model near the singular set $\overline{\mathcal{S}}$. This means that we only have to consider two cases, i.e., when the stabilizer is a circle or when it is a torus. We refer to these sets as $\overline{\mathcal{S}}_1$ and $\overline{\mathcal{S}}_2$, respectively.

5.4.1.1 Blow-up analysis at $\overline{\mathcal{S}}_1$

Let $p \in \overline{\mathcal{S}}_1$ and let U_1 the generator of the stabilizer at p inside $\mathbb{T}^3$. The complement is assumed to be spanned by U_2, U_3. We pick normal coordinates around p, which we identify as 0, using Lemma 2.2.15. We are now, in the set-up of Section 2.2.3 and we deduce that $\tilde{U}_1 = U_1$ and $\tilde{U}_2 = U_2(0), \tilde{U}_3 = U_3(0)$ constant vector fields. If we write $\mathbb{R}^7$ as $\mathbb{R}^3 \oplus \mathbb{C}^2$, where $\mathbb{R}^3$ is determined by $\tilde{U}_2, \tilde{U}_3, \tilde{U}_2 \times_{\varphi_0} \tilde{U}_3$, then $\tilde{U}_1$ generates a $\mathrm{U}(1)$-action on the $\mathbb{C}^2$-component preserving φ_0. Since this $\mathrm{U}(1)$ is a subgroup of G_2 and commutes with $\tilde{U}_2, \tilde{U}_3$ and $\tilde{U}_2 \times_{\varphi_0} \tilde{U}_3$, it acts as a maximal torus in $\mathbb{C}^2$. We conclude that the integral curves of $\nabla^0 \mu_1^0$ passing through p generate, under the limit of the $\mathbb{T}^3$-action, a multiplicity-1 plane. Here, ∇^0 denotes the flat covariant derivative on $\mathbb{R}^7$ and μ_1^0 is the multi-moment map defined by:

$$d\mu_1^0 = *\varphi_0(\tilde{U}_1, \tilde{U}_2, \tilde{U}_3, \cdot).$$

5.4.1.2 Blow-up analysis at $\overline{\mathcal{S}}_2$

Given $p \in \overline{\mathcal{S}}_2$, we denote by U_2, U_3 the generators of the stabilizer of the $\mathbb{T}^3$-action at p and by U_1 the generator of the complement in the Lie algebra of $\mathfrak{t}^3$. Now, we pick normal coordinates at $p = 0$, as above. In particular, we deduce from Section 2.2.3 that $\tilde{U}_1^t \to \tilde{U}_1 = U_1(0)$, constant vector field, and that $\tilde{U}_2 = U_2$, $\tilde{U}_3 = U_3$. We write $\mathbb{R}^7 = \mathbb{R} \times \mathbb{C}^3$, where $\mathbb{R}$ is determined by the flow of $\tilde{U}_1$, and we observe that $\tilde{U}_2, \tilde{U}_3$ generate a $\mathbb{T}^2$, φ_0-preserving action that commutes with $\tilde{U}_1$. Hence, it acts only on the $\mathbb{C}^3$-component as a subset of $\mathrm{SU}(3)$. It is straightforward to see that integral curves of $\nabla^0 \mu_1^0$ passing through p generate, under the limit of the $\mathbb{T}^3$-action, the multiplicity-1 cone: $\mathbb{R} \times N$, where N is the Harvey–Lawson cone in $\mathbb{C}^3$.

Theorem 5.4.5 (Aslan–T. [6]). *Let Σ be a $\mathbb{T}^3$-invariant $*\varphi$-calibrated current of M. Then, Σ is smooth at each point of M where the stabilizer of the $\mathbb{T}^3$-action is 0-dimensional or 1-dimensional. Otherwise, the stabilizer is 2-dimensional and Σ has a tangent cone modelled on the product of the Harvey and Lawson cone with a line.*

Proof. Let Σ be a $*\varphi$-calibrated current which is invariant under the $\mathbb{T}^3$-action. It is clear from the local existence and uniqueness theorem that at each point where the stabilizer of the $\mathbb{T}^3$-action is 0-dimensional, then, Σ is smooth there. In particular, Σ can develop singularities only at $\overline{\mathcal{S}}$.

Note that Σ can not be contained in $\overline{\mathcal{S}}$ and corresponds to an integral curve of $\nabla\mu_1$ in $M \setminus \overline{\mathcal{S}}$. Without loss of generality, we consider a connected component of Σ in $M \setminus \overline{\mathcal{S}}$.

Let $p \in (\operatorname{supp}\Sigma) \cap \overline{\mathcal{S}}$ and let $B_2(0)$ be a neighbourhood of p, identified with 0, as in Lemma 2.2.15. Note that the restriction of Σ to $B_2(0)\setminus\overline{\mathcal{S}}$ corresponds to a unique integral curve of $\nabla\mu_1$ up to picking $B_2(0)$ small enough. Otherwise, $\mu_1\big|_L$ would have an interior maximum or a minimum contradicting Lemma 5.4.2. In particular, the support of the integral curve can not be a loop passing through p. This means that γ_1 as in Fig. 5.2 can not be an integral curve of $\nabla\mu_1$.

We now want to show that, under a suitable blow-up, γ converges to an integral curve of $\nabla^0\mu_1^0$ passing through zero. We can then conclude by the analysis of the local models (cfr. Section 5.4.1.1, Section 5.4.1.2) and by Theorem 2.2.12.

Since $0 \in \overline{\mathrm{Im}\gamma}$, we can choose a sequence of points of $\mathrm{Im}\gamma$: $x_k \in C_k := S_{1/k}(0) = \{x \in B_2(0) : |x|_{\mathbb{R}^7} = \frac{1}{k}\}$. In particular, $kx_k \in S_1(0)$ will converge, up to passing to a subsequence, to some $\overline{x} \in S_1(0)$. We denote by γ_t^x the integral curve of $\widetilde{(\nabla\mu_1)}^t$ with initial value x. Since for $k \to \infty$ we have that $kx_k \to \bar{x}$ and $\widetilde{(\nabla\mu_1)}^t \to \nabla^0\mu_1^0$ because of Lemma 2.2.14, it follows from the theory of ODEs that $\gamma_{1/k}^{kx_k}$ converges to $\gamma_0^{\overline{x}}$ integral curve of $\nabla^0\mu_1^0$ of initial value $\overline{x}$. From the choice of x_k and Lemma 2.2.14, we deduce that $\{\gamma_{1/k}^{kx_k}\}_{k=1}^{\infty}$ is a blow-up of γ and we can conclude the proof. □

Remark 5.4.6. In Section 5.5.3, we will see that there are examples of singular $\mathbb{T}^3$-invariant coassociatives.

Remark 5.4.7. Observe that we have not used the fact that $\mathbb{T}^3$ is a subgroup of $\mathbb{T}^2 \times \mathrm{SU}(2)$. In particular, Theorem 5.4.5 holds in G_2-manifolds with a structure-preserving $\mathbb{T}^3$-action.

On $B := M_P/G$ the $\mathbb{T}^3$-invariant coassociatives correspond to the level sets of ν.

Theorem 5.4.8 (Aslan–T. [6]). *Let Σ_0 be a $\mathbb{T}^3$-invariant coassociative submanifold of M_P. Then, the projection of Σ_0 to B is contained in a level set of ν. Conversely, every level set of ν on B can be lifted to an S^2 of $\mathbb{T}^3$-invariant coassociatives.*

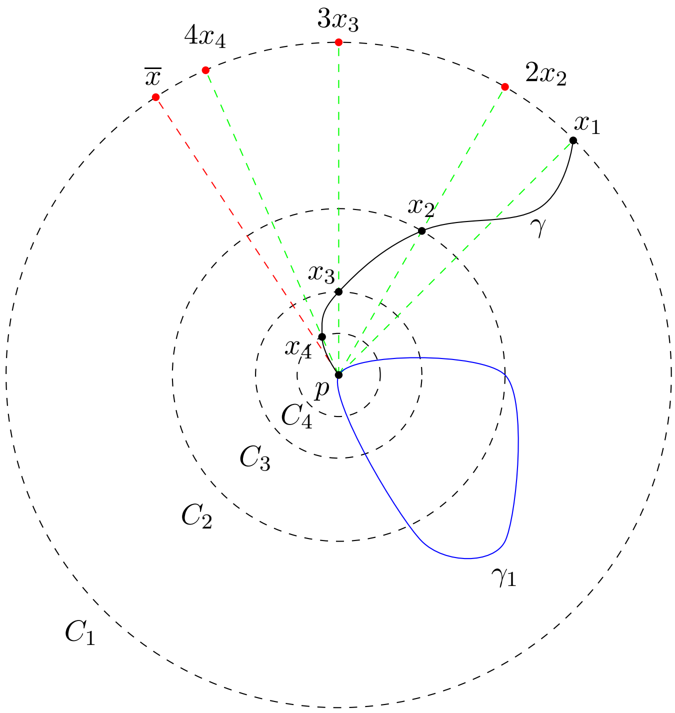


Figure 5.2: Blow-up procedure of Theorem 5.4.5

Proof. If we consider the projection of Σ_0 to $M_P/\mathbb{T}^2$, we obtain a surface $\Sigma_0/\mathbb{T}^2$ which is invariant under the action of an $S^1 \subset G^{\mathrm{SU}(2)}$. So, projecting it to B reduces the dimension to one and we obtain a curve in B. From Proposition 5.4.3 and dimensional reasons, we conclude the proof of this direction. The converse follows from the fact that $\mathbb{T}^3$-invariant coassociatives are in 1-to-1 correspondence with the S^1-reductions of the $G^{\mathrm{SU}(2)}$-bundle $M_P/\mathbb{T}^2$, for a fixed $S^1 \subset G^{\mathrm{SU}(2)}$. □

Remark 5.4.9. Observe that, if Σ_0 is a $\mathbb{T}^3$-invariant coassociative with respect to some $\mathbb{T}^2 \times S^1 \subset \mathbb{T}^2 \times \mathrm{SU}(2)$ and its projection to B is contained in a level set of ν, then, $g \cdot \Sigma_0$ is also a $\mathbb{T}^3$-invariant coassociative with respect to $\mathbb{T}^2 \times (g \cdot S^1) \subset \mathbb{T}^2 \times \mathrm{SU}(2)$ and projects to the same level set of ν.

As a consequence of this discussion we deduce that B has a nice parametrization determined by associative and coassociative submanifolds, which are $\mathbb{T}^2$-invariant and $\mathbb{T}^3$-invariant respectively.

Corollary 5.4.10 (Associative/coassociative parametrization of the quotient)**.** *Consider the local orthogonal parametrization of $B := M_P/G$ given by $(|\mu|, \nu)$. Then, the coordinate lines correspond to $\mathbb{T}^2$-invariant associative submanifolds and $\mathbb{T}^3$-invariant coassociative submanifolds, respectively.*

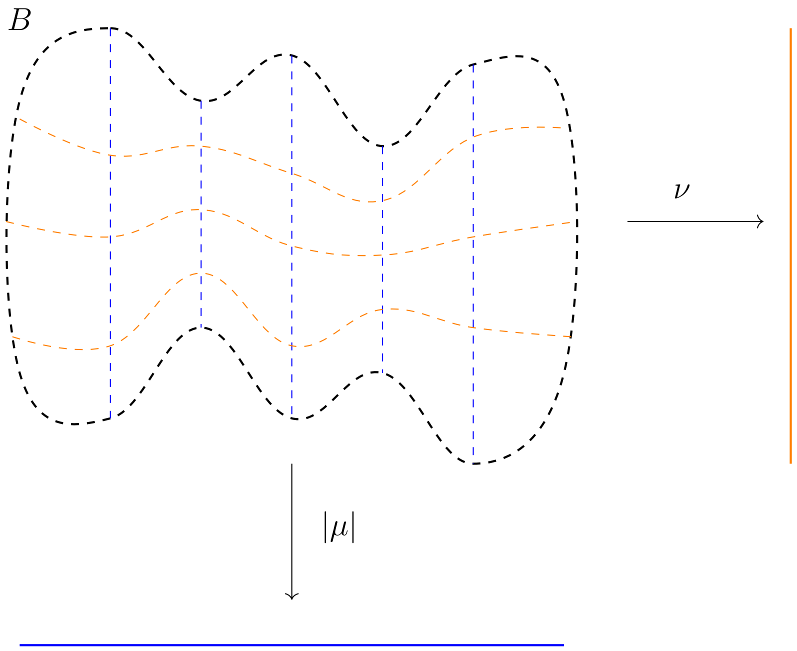


Figure 5.3: Associative/coassociative parametrization of B

Proof. The proof follows immediately from Theorem 5.3.5 and Theorem 5.4.8. □

5.4.2 SU(2)-invariant coassociative submanifolds

For the sake of brevity we omit the proofs, which are analogous to the other cases. In order to guarantee the existence of SU(2)-invariant coassociatives, we need to assume that $\varphi(V_1, V_2, V_3) \equiv 0$ from now on. Actually, it is enough to have that it vanishes at a point. Indeed, Cartan's formula, together with $[\mathcal{L}_X, i_Y] = i_{[X,Y]}$, implies that $\varphi(V_1, V_2, V_3)$ is a constant function. A sufficient condition, but not necessary as shown in Section 5.5.3.5, is that the $\mathrm{SU}(2)/\Gamma_2$ action has a singular orbit. We denote the singular set of this action by $\tilde{S}$.

Proposition 5.4.11. *Let Σ_0 be a $\mathrm{SU}(2)$-invariant coassociative submanifold of $M \setminus \tilde{\mathcal{S}}$. Then, $\Sigma_0/\mathrm{SU}(2)$ is an integral curve of the nowhere vanishing vector field $\nabla\eta$ in $(M \setminus \tilde{\mathcal{S}})/\mathrm{SU}(2)$. Conversely, every integral curve of $\nabla\eta$ in $(M \setminus \tilde{\mathcal{S}})/\mathrm{SU}(2)$ is the projection of a $\mathrm{SU}(2)$-invariant coassociative in $M \setminus \tilde{\mathcal{S}}$.*

Lemma 5.4.12. *Let γ be an integral curve of $\nabla\eta$ in $M \setminus \tilde{\mathcal{S}}$. Then, the multi-moment map η is strictly increasing along γ.*

Proposition 5.4.13. *The flow of $\nabla\eta$ preserves the orbit type of G. Hence, the integral curves of $\nabla\eta$ stay in the same strata of the stratification described in Theorem 5.1.7.*

By Lemma 5.1.5, the action of $\mathbb{T}^2$ on M induces on the quotient $M_P/(\mathrm{SU}(2)/\Gamma_2)$ a $G^{\mathbb{T}^2}$ principal bundle structure with base space B. Let $\mathcal{H}$ be a connection on $M_P/(\mathrm{SU}(2)/\Gamma_2)$ such that the $\mathbb{T}^2$-invariant vector field $\nabla\eta$ is horizontal. For instance, the connection induced by the metric g_φ satisfies this property $0 = *\varphi(U_i, V_1, V_2, V_3) = g(U_i, \nabla\eta)$ for $i = 1, 2$. As in Theorem 5.3.5, we deduce the following proposition.

Theorem 5.4.14 (Aslan–T. [6]). *Let $\mathcal{H}$ be a connection on the principal $G^{\mathbb{T}^2}$-bundle $M_P/\mathrm{SU}(2) \to B$ such that $\nabla\eta \in \mathcal{H}$. Let γ be a curve in $M_P/(\mathrm{SU}(2)/\Gamma_2)$. The following are equivalent:*

1. *The pre-image $\pi^{-1}_{\mathrm{SU}(2)}(\mathrm{im}\gamma)$ is a $\mathrm{SU}(2)$ invariant co-associative in M_P,*
2. *γ is an integral curve of $\nabla\eta$,*
3. *γ is the horizontal lift of an integral curve of $\nabla\eta$ in B.*

Moreover, the correspondence between (1) and (2) is 1-to-1, while for every integral curve of $\nabla\eta$ in B there is a $\mathbb{T}^2$-family of integral curves of $\nabla\eta$ on $M_P/(\mathrm{SU}(2)/\Gamma_2)$.

Remark 5.4.15. Note that, we can not conclude that we have an SU(2)-invariant coassociative fibration in the sense of Definition 3.1.5. Indeed, Theorem 5.4.14 only implies that M_P admits a foliation of coassociative leaves.

Differently from the other cases, the obvious 1-forms that would give constant quantities on SU(2)-invariant coassociatives are not closed. These are defined as:

$$\omega_1 := \varphi(V_2, V_3, \cdot), \quad \omega_2 := \varphi(V_3, V_1, \cdot), \quad \omega_3 := \varphi(V_1, V_2, \cdot). \tag{5.4.1}$$

Remark 5.4.16. These 1-forms can be put in the context of weak homotopy moment-maps (see [39] and references therein). Moreover, since $i_{U_l}\omega_i = -\theta_i^l$ the ω_is do not descend to the quotients: $M_P/(\mathrm{SU}(2)/\Gamma_2), M_P/\,\mathbb{T}^2$ and B.

Proposition 5.4.17. *A 4-dimensional submanifold, Σ_0, is a $\mathrm{SU}(2)$-invariant coassociative submanifold of $M \setminus \tilde{\mathcal{S}}$ if and only if $\omega^i\big|_{\Sigma_0} = 0$ for all $i = 1, 2, 3$.*

Remark 5.4.18. The previous proposition does not use the additional $\mathbb{T}^2$-action. In particular, we re-obtain the characterizing ODEs for the SU(2)-invariant coassociative submanifolds on the Bryant–Salamon manifold $\Lambda^2_-(S^4)$ and $\Lambda^2_-(\mathbb{CP}^2)$ computed in [49].

In a similar fashion to Theorem 5.4.5, one can obtain the following regularity result on SU(2)-invariant coassociative submanifolds.

Theorem 5.4.19 (Aslan–T. [6]). *Every* SU(2)*-invariant* $*\varphi$*-calibrated current in* M *is a smooth submanifold.*

Remark 5.4.20. The existence of the $\mathbb{T}^2$-action is crucial for Theorem 5.4.19. Indeed, Karigiannis and Lotay constructed in [49] examples of asymptotically singular SU(2)-invariant coassociatives on $\Lambda^2_-(S^4)$ and on $\Lambda^2_-(\mathbb{CP}^2)$.

5.5 Examples

In this final section, we consider the flat space, $\mathbb{C}^3 \times S^1$, the G_2 manifolds constructed by Foscolo–Haskins–Nordström in [32] and the Bryant–Salamon G_2 manifolds of topology $S^3 \times \mathbb{R}^4$. On these spaces we explicitly discuss the general theory we developed in the previous sections.

5.5.1 Flat $\mathbb{C}^3 \times S^1$

Given any Calabi–Yau structure on a six-manifold M there is natural G_2-structure on $S^1 \times M$ given by

$$\varphi := \mathrm{Re}\Omega - d\theta \wedge \omega, \quad *\varphi = -\frac{1}{2}\omega^2 - d\theta \wedge \mathrm{Im}\Omega \tag{5.5.1}$$

where θ parametrizes S^1.

Consider the flat Calabi–Yau structure on $\mathbb{C}^3$. Namely, if $(z_1, z_2, z_3) \in \mathbb{C}^3$ is such that $z_j = x_j + iy_j$, then:

$$\omega := \sum_{j=1}^{3} dx_j \wedge dy_j$$

is the standard Kähler form of $\mathbb{C}^3$ and

$$\Omega := dz_1 \wedge dz_2 \wedge dz_3$$

is the standard holomorphic volume form. The induced G_2-structure on $\mathbb{C}^3 \times S^1$ is the flat one.

Clearly, $\mathbb{C}^3 \times S^1$ admits the required symmetry, where $(e^{i\lambda}, e^{it}, A) \in U(1) \times U(1) \times \mathrm{SU}(2)$ acts on $\mathbb{C}^3 \times S^1$ as follows:

$$(e^{it}, e^{i\lambda}, A)(z_1, z_2, z_3, e^{i\theta}) \mapsto (e^{2i\lambda} z_1, e^{-i\lambda} A \cdot (z_2, z_3)^T, e^{i(t+\theta)}).$$

Associatives in $\mathbb{C}^3 \times S^1$ with $\mathbb{T}^2$-invariance are products of S^1 with an holomorphic curve in $\mathbb{C}^3$, invariant under the remaining S^1. These are exactly the fibres of the map

$$\mathbb{C}^3 \to \mathbb{C}^2, \quad (z_1, z_2, z_3) \mapsto (z_1 z_2^2, z_1 z_3^2),$$

where the singular set is mapped to 0. In the following, we describe the fibres using our moment map method.

It is straightforward to verify that the stratification of Section 5.1.1 is as follows:

- $M_P = \mathbb{C} \setminus \{0\} \times \mathbb{C}^2 \setminus \{\underline{0}\} \times S^1$,
- $\mathcal{S}_1 = \{0\} \times \mathbb{C}^2 \setminus \{\underline{0}\} \times S^1$,
- $\mathcal{S}_2 = \emptyset$,
- $\mathcal{S}_3 = \mathbb{C} \setminus \{0\} \times \{\underline{0}\} \times S^1$,
- $\mathcal{S}_4 = \{0\} \times \{\underline{0}\} \times S^1$.

The generators of the $\mathfrak{t}^2$ component are:

$$U_1 = \partial_\theta \quad U_2 = 2(-y_1\partial_{x_1} + x_1\partial_{y_1}) + y_2\partial_{x_2} - x_2\partial_{y_2} + y_3\partial_{x_3} - x_3\partial_{y_3},$$

while the generators of the $\mathfrak{su}(2)$ component are:

$$\begin{aligned}
V_1 &= \frac{1}{2}(-y_3\partial_{x_2} + x_3\partial_{y_2} - y_2\partial_{x_3} + x_2\partial_{y_3}),\\
V_2 &= \frac{1}{2}(-x_3\partial_{x_2} - y_3\partial_{y_2} + x_2\partial_{x_3} + y_2\partial_{y_3}),\\
V_3 &= \frac{1}{2}(y_2\partial_{x_2} - x_2\partial_{y_2} - y_3\partial_{x_3} + x_3\partial_{y_3}).
\end{aligned}$$

From these, we compute the multi-moment maps ν, η:

$$\nu = |z_1|^2 - \frac{1}{2}(|z_2|^2 + |z_3|^2), \quad \eta = \frac{1}{32}(|z_2|^2 + |z_3|^2)^2,$$

and μ, θ:

$$\mu = \frac{1}{2}\begin{pmatrix} \mathrm{Im}(z_1(z_2^2 - z_3^2)) \\ -\mathrm{Re}(z_1(z_2^2 + z_3^2)) \\ 2\mathrm{Im}(z_1 z_2 z_3) \end{pmatrix},$$

$$\theta^1 = \frac{1}{4}\begin{pmatrix} 2\mathrm{Re}(z_2\overline{z}_3) \\ 2\mathrm{Im}(z_2\overline{z}_3) \\ (|z_3|^2 - |z_2|^2), \end{pmatrix} \quad \theta^2 = \frac{1}{2}\begin{pmatrix} \mathrm{Re}(z_1(z_2^2 - z_3^2)) \\ \mathrm{Im}(z_1(z_2^2 + z_3^2)) \\ 2\mathrm{Re}(z_1 z_2 z_3) \end{pmatrix}.$$

Since the metric is Euclidean, it is easy to compute the gradients:

$$\begin{aligned}
\nabla\nu &= U_1 \times U_2 = 2(x_1\partial_{x_1} + y_1\partial_{y_1}) - (x_2\partial_{x_2} + y_2\partial_{y_2} + x_3\partial_{x_3} + y_3\partial_{y_3}),\\
\nabla\eta &= -\frac{1}{8}(|x_2|^2 + |y_2|^2 + |x_3|^2 + |y_3|^2)(x_2\partial_{x_2} + y_2\partial_{y_2} + x_3\partial_{x_3} + y_3\partial_{y_3}).
\end{aligned}$$

We identify the principal set $M_P = \mathbb{C} \setminus \{0\} \times \mathbb{C}^2 \setminus \{\underline{0}\} \times S^1$ with $\mathbb{C} \setminus \{0\} \times \mathbb{H} \setminus \{0\} \times S^1$. Using polar coordinates $(re^{i\xi}, \rho q) \in \mathbb{C} \setminus \{0\} \times \mathbb{H} \setminus \{0\}$ for $r, \rho \in \mathbb{R}^+$ and $\xi \in S^1$ and $q \in \mathrm{Sp}(1)$ we get

$$U_1 \times U_2 = 2r\partial_r - \rho\partial_\rho.$$

In particular, the integral curves of $U_1 \times U_2$ can be computed explicitly. They induce, for any fixed constant $C \in \mathbb{R}^+$ and $q \in \mathrm{Sp}(1)$, the following $\mathbb{T}^2$-invariant associatives:

$$L_{C,q} = \left\{ \left(re^{2it}, \frac{C}{r^{1/2}} e^{-it} q, e^{i\theta} \right) : r \in \mathbb{R}^+, t \in S^1, \theta \in S^1 \right\}, \tag{5.5.2}$$

which have topology $\mathbb{R} \times \mathbb{T}^2$. If we write $q = a_0 + ia_1 + ja_2 + ka_3$, then e^{-it} acts on q not by quaternionic left multiplication but, after the identification with $\mathbb{C}^2$, i.e. $q = (a_0 + ia_1, a_2 + ia_3) \in \mathbb{C}^2$.

Theorem 5.5.1 (Aslan–T. [6]; $\mathbb{T}^2$-invariant associatives in $\mathbb{C}^3 \times S^1$)**.** *Consider the stratification of $\mathbb{C}^3 \times S^1$ into $M_P \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$, as given in Section 5.1.1. Then, each strata decomposes into $\mathbb{T}^2$-invariant associatives in the following way:*

- *M_P can be fibred by $\mathbb{T}^2$-invariant associatives of topology $\mathbb{T}^2 \times \mathbb{R}$ that are defined by Eq.* (5.5.2)*.*
- *$\mathcal{S}_1 = \{0\} \times \mathbb{C}^2 \setminus \{\underline{0}\} \times S^1$ is fibred by totally geodesic $\mathbb{T}^2$-invariant associatives of topology $S^1 \times (\mathbb{R}^2 \setminus \{0\})$ via the Hopf fibration map. Clearly, these associative extend to smooth associatives of topology $S^1 \times \mathbb{R}^2$.*
- *$\mathcal{S}_2 = \emptyset$.*
- *$\mathcal{S}_3 = \mathbb{C} \setminus \{0\} \times \{\underline{0}\} \times S^1$ is clearly an associative which extends to a smooth associative of topology $S^1 \times \mathbb{R}^2$ if we add $\mathcal{S}_4 = \{0\} \times \{\underline{0}\} \times S^1$ to it.*

In particular, this decomposition defines a fibration in the sense of Definition 3.1.5.

To put the fibration in the context of Section 5.3.3, we can see that $M_P/\mathbb{T}^2 = \{(r, \rho, q)\} \cong \mathbb{R}^+ \times \mathbb{R}^+ \times \mathrm{Sp}(1)$ and, hence, the base of the $\mathrm{SU}(2)$-bundle B is given by $\{(r, \rho)\} \cong \mathbb{R}^+ \times \mathbb{R}^+$. The multi-moment maps $(|\mu|, \nu)$ in this coordinates become:

$$\nu = 2r^2 - \rho^2, \quad \mu = \begin{pmatrix} 2r\rho^2(a_0a_1 - a_2a_3) \\ r\rho^2(a_0^2 - a_1^2 + a_2^2 - a_3^2) \\ 2r\rho^2(-a_0a_3 - a_1a_2) \end{pmatrix},$$

which satisfy the conditions of Corollary 5.3.10 as $|\mu| = r\rho^2$. In particular, under a suitable identification of $\mathbb{C}^2 \cong \mathbb{H}$, the above trivialization is such that the $\mathrm{SU}(2)$-component of $U_1 \times U_2$ identically vanishes.

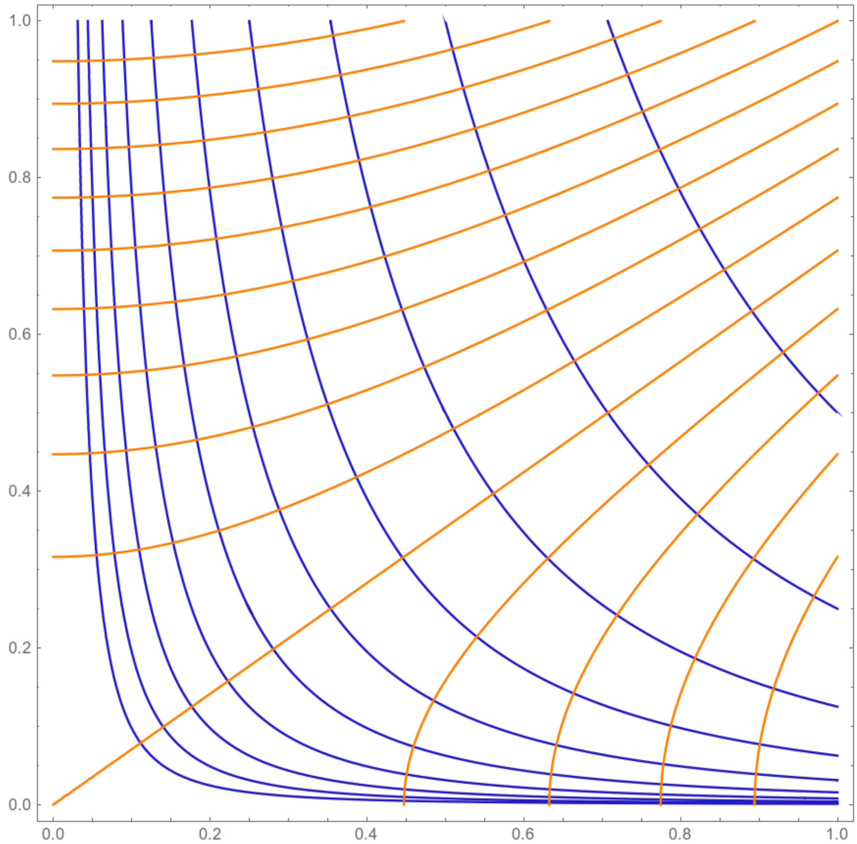


Figure 5.4: Blue: The level sets of $|\mu| = r\rho^2$ in $B = \mathbb{R}^+ \times \mathbb{R}^+$. Every level set represents an SU(2)-family of $\mathbb{T}^2$-invariant associatives in M_P. Orange: The level sets of $\nu = 2r^2 - \rho^2$. Every level set represents an S^2 family of $\mathbb{T}^3$-invariant coassociatives in M_P.

We now fix the $S^1 \subset \mathrm{SU}(2)$ generated by V_3 and describe the coassociative submanifolds invariant under the resulting $\mathbb{T}^3$-action. These coassociatives are products of S^1 with a $\mathbb{T}^2$-invariant special Lagrangian submanifold of phase $-\pi/2$ in $\mathbb{C}^3$, which are classified in [37, III.3.A Theorem 3.1] as the level sets of $|z_1|^2 - |z_j|^2$ for $j = 2, 3$ and $\mathrm{Re}(z_1 z_2 z_3) = c_3$. This agrees with our moment map description, as the $\mathbb{T}^3$-invariant coassociatives are level sets of $(\theta_3^1, \theta_3^2, \nu)$ on $M \setminus \mathcal{S}_4$.

Finally, observe that every four-plane $\{p\} \times \mathbb{C}^2 \subset S^1 \times \mathbb{C} \times \mathbb{C}^2$ is coassociative for every $p \in S^1 \times \mathbb{C}$. Moreover, it is SU(2)-invariant because SU(2) only acts on the $\mathbb{C}^2$-component. Alternatively, we obtain the same result by computing

$$\nabla \eta = \frac{\rho^3 \partial_\rho}{8}.$$

whose integral curves correspond to SU(2)-invariant coassociatives in M_P.

5.5.2 Foscolo–Haskins–Nordström manifolds

The FHN manifolds, described in Section 2.2.5, admit the required $\mathbb{T}^2 \times \mathrm{SU}(2)$-symmetry under the additional assumption $a := a_2 = a_3$ and $b := a_1$. Indeed, the action of $(\lambda_1, \lambda_2, \gamma) \in \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{SU}(2)$ on $([p, q], t) \in (\mathrm{SU}(2) \times \mathrm{SU}(2))/K_0 \times I$, given as follows:

$$(\lambda_1, \lambda_2, \gamma) \cdot ([p, q], t) = ([\lambda_1 p\overline{\lambda}_2, \gamma q\overline{\lambda}_2], t), \tag{5.5.3}$$

is structure preserving (cfr. Eq. (2.2.8)), where the two U(1)s are generated by quaternionic multiplication by i.

Remark 5.5.2. Obviously, there is another action of $(\lambda_1, \lambda_2, \gamma) \in \mathbb{T}^2 \times \mathrm{SU}(2)$ on $([p, q], t) \in (\mathrm{SU}(2) \times \mathrm{SU}(2))/K_0 \times I$:

$$(\lambda_1, \lambda_2, \gamma) \cdot ([p, q], t) = ([\gamma p\overline{\lambda}_2, \lambda_1 q\overline{\lambda}_2], t).$$

The discussion is analogous to the one for Eq. (5.5.3) and we leave it to the reader.

5.5.2.1 The stratification

We first deal with the set: $(\mathrm{SU}(2)\times\mathrm{SU}(2))/K_0\times\mathrm{Int}(I)$. If K_0 is trivial, it is straightforward to see that the principal stabilizer of the $\mathbb{T}^2 \times \mathrm{SU}(2)$-action is generated by $(-1_{\mathbb{T}^2}, -1_{\mathrm{SU}(2)})$. On the other hand, if $K_0 = K_{m,n} \cap K_{2,-2}$ the principal stabilizer is a discrete subgroup of $\mathbb{T}^2 \times \mathrm{SU}(2)$ with $\Gamma_1 \neq 0$. In both cases, $G^{\mathrm{SU}(2)} = \mathrm{SO}(3)$ and the singular set is given by:

$$\mathcal{S}_+ = \{([p, q], t) \in (\mathrm{SU}(2) \times \mathrm{SU}(2))/K_0 \times \mathrm{Int}(I) : p \in \mathbb{C} \times \{0\} \subset \mathbb{H}\},$$
$$\mathcal{S}_- = \{([p, q], t) \in (\mathrm{SU}(2) \times \mathrm{SU}(2))/K_0 \times \mathrm{Int}(I) : p \in \{0\} \times \mathbb{C} \subset \mathbb{H}\},$$

with 1-dimensional stabilizer. If K_0 is trivial, the stabilizer at $([p, q], t)$ is either the circle $\{(\lambda, \lambda, q\lambda\overline{q})\}$ or $\{(\lambda, \overline{\lambda}, q\overline{\lambda}\overline{q})\}$, depending if $([p, q], t)$ is in $\mathcal{S}_+$ or $\mathcal{S}_-$.

To understand the stratification on $(\mathrm{SU}(2) \times \mathrm{SU}(2))/K$ we need to distinguish three cases:

Case 1 ($K = \Delta\,\mathrm{SU}(2)$). If we identify $\mathrm{SU}(2) \times \mathrm{SU}(2)/\Delta\,\mathrm{SU}(2)$ with S^3 via $[(p, q)] \mapsto p\overline{q}$, then, the action of $\mathbb{T}^2 \times \mathrm{SU}(2)$ becomes, for every $p \in S^3 \subset \mathrm{Sp}(1)$:

$$(\lambda_1, \lambda_2, \gamma) \cdot p = \lambda_1 p\overline{\gamma}.$$

We deduce that the stabilizer is always 2-dimensional and it is the two torus: $\{(\lambda_1, \lambda_2, \overline{p}\lambda_1 p)\}$.

Case 2 ($K = \{1_{\mathrm{SU}(2)}\} \times \mathrm{SU}(2)$). Under the identification of $(\mathrm{SU}(2) \times \mathrm{SU}(2))/K$ to S^3 given by $[(p, q)] \mapsto p$, the $\mathbb{T}^2 \times \mathrm{SU}(2)$ action becomes:

$$(\lambda_1, \lambda_2, \gamma) \cdot p = \lambda_1 p\overline{\lambda_2},$$

where $p \in S^3 \cong \mathrm{Sp}(1)$. Hence, the stabilizer is the $\mathbb{Z}_2 \times \mathrm{SU}(2)$ given by $\{\pm 1_{\mathbb{T}^2}, \gamma\}$ if $p \notin (\mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}) \subset \mathrm{Sp}(1)$, otherwise it is the 4-dimensional $\mathrm{SU}(2) \times \mathrm{U}(1)$ given by $\{(\lambda, \overline{\lambda}, \gamma)\}$ or $\{(\lambda, \lambda, \gamma)\}$.

Case 3 ($K = K_{m,n}$). Under the isomorphism for $K_{m,n}$ of Eq. (2.2.5), we have that two elements of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ are in the same equivalence class if and only they they are equal up to right multiplication of $(e^{-in\theta}, e^{im\theta})$ for some $\theta \in [0, 2\pi)$. It is straightforward to verify that the stabilizer at $[(p,q)]$ is 1-dimensional if $p \notin \mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C} \subset \mathrm{Sp}(1)$. Otherwise, it is 2-dimensional.

5.5.2.2 The multi-moment maps

In this subsection we compute the multi-moment maps on $(\mathrm{SU}(2) \times \mathrm{SU}(2))/K_0 \times \mathrm{Int}(I)$ and hence, by continuity, on the whole space.

Consider the Hopf fibration map $S^3 \subset \mathbb{H} \to S^2 \subset \mathrm{im}\mathbb{H}$ that maps $p \to \overline{p}ip$. Taking two copies of the Hopf fibration, together with the identity on $\mathrm{Int}(I)$, yields the quotient map to the $\mathbb{T}^2$-quotient:

$$\begin{aligned}\pi_{\mathbb{T}^2} : (\mathrm{SU}(2) \times \mathrm{SU}(2))/K_0 \times \mathrm{Int}(I) &\to S^2 \times S^2 \times \mathrm{Int}(I)\\ (p,q,t) &\mapsto (v,w,t),\end{aligned}$$

where $v = q\overline{p}ip\overline{q} = v_1 i + v_2 j + v_3 k$ and $w = qi\overline{q} = w_1 i + w_2 j + w_3 k$.

If $h := \overline{p}ip = h_1 i + h_2 j + h_3 k$ and $g_l := \overline{q}lq = g_{l,1} i + g_{l,2} j + g_{l,3} k$, then, the Killing vector fields of the $\mathbb{T}^2 \times \mathrm{SU}(2)$-action satisfying Eq. (5.1.2) are:

$$\begin{aligned}U_1(p,q,r) &= (ip,0,0) = (p\overline{p}ip,0,0) = -\sum_{m=1}^{3} h_m e_m(p,q,r),\\ U_2(p,q,r) &= (-pi,-qi,0) = e_1 + f_1,\\ V_l(p,q,r) &= -\frac{1}{2}(0,-lq,0) = -\frac{1}{2}(0,q\overline{q}lq,0) = \frac{1}{2}\sum_{m=1}^{3} g_{l,m} f_m,\end{aligned}$$

where $l = 1,2,3$ and e_l, f_l form the standard orthonormal left invariant frame of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ as defined in Section 2.2.5.2.

A straightforward computation gives the multi-moment maps in the quotient:

$$\begin{aligned}\nu &= -4(b - c_1)\langle v, w\rangle_{\mathbb{R}^3}, & \mu &= -4\dot{a}bv \times_{\mathbb{R}^3} w,\\ \theta^1 &= 2av - 2(a-b)\langle v, w\rangle_{\mathbb{R}^3} w, & \theta^2 &= -2(b + c_2)w,\\ \eta &= \text{Primitive of}\left(\frac{2ba^2 + c_2(b^2 + 2a^2 + c_1c_2)}{\sqrt{-\Lambda}}\right),\end{aligned} \tag{5.5.4}$$

where we used the following identities:

$$h_1 = \langle v, w\rangle_{\mathbb{R}^3}, \quad \langle h, g_l\rangle_{\mathbb{R}^3} = v_l, \quad g_{l,1} = w_l, \quad (h \times g_l)_1 = (v \times w)_l.$$

5.5.2.3 Associatives in the singular set

As a first step, we deal with $(\mathrm{SU}(2) \times \mathrm{SU}(2))/K_0 \times \mathrm{Int}(I)$. Observe that the images of $\mathcal{S}_+$ and $\mathcal{S}_-$ under the $\mathbb{T}^2$-projection map $\pi_{\mathbb{T}^2}$ are:

$$\mathcal{O}_+ = \{(v, v, t) \in S^2 \times S^2 \times \mathrm{Int}(I)\}, \quad \mathcal{O}_- = \{(v, -v, t) \in S^2 \times S^2 \times \mathrm{Int}(I)\}.$$

As argued in Lemma 5.1.5, the action of $G^{\mathrm{SU}(2)}$ descends to $(M \setminus \mathcal{S})/\mathbb{T}^2$ and $G^{\mathrm{SU}(2)} = \mathrm{SO}(3)$ acts diagonally on $S^2 \times S^2$. This SO(3)-action is of cohomogeneity one and the singular orbits are $\mathcal{O}_+$ and $\mathcal{O}_-$ which have stabilizer diffeomorphic to S^1.

The proof of Theorem 5.3.9 contains the construction of a fibration $\mathcal{S}_1 \to S^2$ with associative fibres. These are zero sets of Killing vector fields. For $\mathcal{S}_+ \cup \mathcal{S}_-$, the fibration can be described explicitly as follows.

Let $u \colon (\mathrm{SU}(2) \times \mathrm{SU}(2))/K_0 \times \mathrm{Int}(I) \to S^2 \times S^2$ be the composition of $\pi_{\mathbb{T}^2}$ with the projection $p : S^2 \times S^2 \times \mathrm{Int}(I) \to S^2 \times S^2$. Then, u maps $\mathcal{S}_+ \cup \mathcal{S}_-$ to $p(\mathcal{O}_+) \cup p(\mathcal{O}_-)$ and the fibres are associative.

Proposition 5.5.3. *The map $u \colon \mathcal{S}_+ \cup \mathcal{S}_- \to p(\mathcal{O}_+) \cup p(\mathcal{O}_-) \cong S^2 \cup S^2$ is a submersion with totally geodesic $\mathbb{T}^2$-invariant associative fibres of topology $\mathbb{T}^2 \times \mathrm{Int}(I)$.*

Proof. By SU(2)-equivariance, it suffices to show the statement for a single fibre in each of $\mathcal{O}_+$ and $\mathcal{O}_-$. We restrict ourselves to the fibre over the point $\{(i, i)\} \in \mathcal{O}_+ \subset \mathrm{Im}\mathbb{H} \times \mathrm{Im}\mathbb{H}$, as the $\mathcal{O}_-$ case is analogous.

Note that

$$u^{-1}(\{(i, i)\}) = \{([p, q], t) : p, q \in (\mathbb{C} \times \{0\}) \cap \mathrm{Sp}(1), t \in \mathrm{Int}(I)\},$$

which is the fixed set of the involution $(i, i, i) \in U(1) \times U(1) \times \mathrm{Sp}(1)$ acting on $(\mathrm{SU}(2) \times \mathrm{SU}(2))/K_0 \times \mathrm{Int}(I)$ as in Eq. (5.5.3). So $u^{-1}(\{(i, i)\})$ is a connected component of the fixed set of (i, i, i), which is therefore totally geodesic and associative. □

We now consider the singular orbit $\mathrm{SU}(2) \times \mathrm{SU}(2)/K$. If $K = \Delta\,\mathrm{SU}(2)$ or $K = \{1\} \times \mathrm{SU}(2)$, then $\mathrm{SU}(2) \times \mathrm{SU}(2)/K$ is an associative submanifold because it is either $\mathcal{S}_2$ or $\mathcal{S}_3 \cup \mathcal{S}_4$. For $K = K_{m,n}$, the singular orbit, $\mathrm{SU}(2) \times \mathrm{SU}(2)/K_{m,n}$, is diffeomorphic to $S^3 \times S^2$ and it admits a submersion onto S^2:

$$F : (\mathrm{SU}(2) \times \mathrm{SU}(2))/K_{m,n} \to S^2 \quad [(p, q)] \mapsto qi\overline{q},$$

with fibres that are $\mathbb{T}^2$-invariant associative submanifolds, of topology the lens space: $L(m; -n, n)$.

In order to prove the previous claim, we observe that, by SU(2)-equivariance, it is enough to show that $F^{-1}(\{i\}) = \{[p,q] : q \in \mathbb{C}\}$ has the desired properties. By inspection, it is straightforward to deduce that it is $\mathbb{T}^2$-invariant and of the given topology. Associativity of $F^{-1}(\{i\})$ follows because it is a connected component of the set with 2-dimensional stabilizer with respect to the action of Remark 5.5.2. Moreover, there are two additional $\mathbb{T}^2$-invariant associative submanifolds in $\mathrm{SU}(2) \times \mathrm{SU}(2)/K_{m,n}$: the two components of $\mathcal{S}_2$ described in the stratification discussion of Section 5.5.2.1, which have topology $L(n; m, -m)$.

Finally, note that for all possible K, the associative submanifolds of Proposition 5.5.3 extend smoothly to associatives of topology $S^1 \times \mathbb{R}^2$ because of Theorem 5.3.11.

5.5.2.4 Associatives in the principal set

On the principal set

$$M_P = ((\mathrm{SU}(2) \times \mathrm{SU}(2)) \times \mathrm{Int}(I)) \setminus (\mathcal{S}_+ \cup \mathcal{S}_-)\,,$$

we are able to give an an explicit parametrization of the $G^{\mathrm{SU}(2)}$-bundle described in Section 5.3.2.

Consider the maps:

$$\Psi\colon\ \mathrm{SO}(3) \times (0,\pi) \to S^2 \times S^2, \quad (g,\theta) \mapsto (g_1, (g_1 \cos\theta - g_2 \sin\theta))$$

where g_1, g_2 and g_3 are the column vectors of g, and:

$$A\colon S^2 \times S^2 \setminus (p(\mathcal{O}_+ \cup \mathcal{O}_-)) \to \mathrm{SO}(3), \quad (v,w) \mapsto \left(\left(v, \frac{1}{\sin\theta}(\cos\theta v - w), -\frac{1}{\sin(\theta)} v \times w\right)\right),$$

where $\theta \in (0,\pi)$ is defined by $\langle v, w\rangle_{\mathbb{R}^3} = \cos\theta$. The map (A,θ) is the inverse of Ψ, and Ψ is a diffeomorphism that is equivariant with respect to the action of SO(3) on both spaces, where SO(3) acts on $\mathrm{SO}(3) \times (0,\pi)$ by left multiplication on the SO(3) factor. The singular orbits $\mathcal{O}_+$ and $\mathcal{O}_-$ are the images of $\{0\} \times \mathrm{SO}(3)$ and $\{\pi\} \times \mathrm{SO}(3)$ if Ψ is extended to $\mathrm{SO}(3) \times [0,\pi]$.

By taking the identity on the component $\mathrm{Int}(I)$ we get the equivariant diffeomorphism, which we also denote by Ψ:

$$\Psi\colon\ \mathrm{SO}(3) \times (0,\pi) \times \mathrm{Int}(I) \to M_P/\mathbb{T}^2 = (S^2 \times S^2 \setminus (p(\mathcal{O}_+) \cup p(\mathcal{O}_-))) \times \mathrm{Int}(I).$$

This means that the base space of the $G^{\mathrm{SU}(2)}$-bundle described in Section 5.3.2 is diffeomorphic to $B = (0,\pi) \times \mathrm{Int}(I)$ and Ψ is a global trivialization of $M_P/\mathbb{T}^2 \to B$. With respect to this trivialization, we have:

$$|\mu| = 4\dot{a}\dot{b} \sin\theta, \quad \nu = -4(b - c_1)\cos\theta.$$

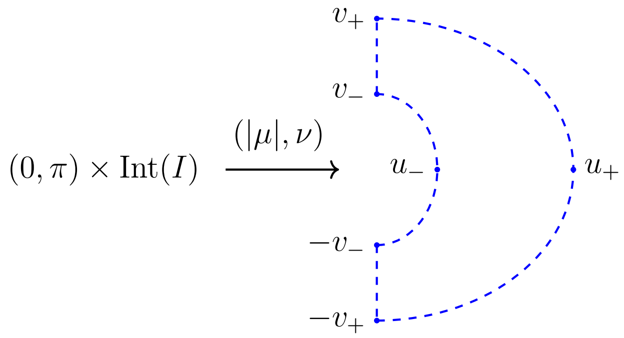


Figure 5.5: Image of α

In order to apply the machinery of Section 5.3.3, we need the following lemma. In our case, we will have $\alpha = (|\mu|, \nu)$, $u = 4\dot{a}\dot{b}$ and $v = \pm 4(b - c_1)$, depending on its sign.

Lemma 5.5.4. *Let u, v be two functions from an interval, $\mathrm{Int}(I)$, to $\mathbb{R}^+$. If $\dot{u}, \dot{v}$ are both positive or both negative everywhere, then, $\alpha(\theta, t) = (u(t)\sin(\theta), v(t)\cos(\theta))$ is a diffeomorphism from $(0, \pi) \times \mathrm{Int}(I)$ onto its image in $\mathbb{R} \times \mathbb{R}^+$. Moreover, if v_- is the infimum of v over I. Then, $(u(t)\cos(\theta))^{-1}(c)$ is connected if $c > u_-$ and has two connected components otherwise. In particular, the map α is a diffeomorphism onto its image and the image is convex if and only if $u_- = 0$.*

Proof. The determinant of the Jacobian vanishes if and only if $\dot{u}v\sin^2(\theta) + \cos^2(\theta)u\dot{v} = 0$, which never happens because $\dot{u}v$ and $\dot{v}u$ have the same sign. So, α is a local diffeomorphism and it remains to show that it is injective. For a fixed value, t_0, of t the function $\alpha(\theta, t_0)$ traces out a half ellipse centred at the origin with semi-axes $u(t_0), v(t_0)$. If t_1 is another fixed value for t, then the ellipses $\alpha(\theta, t_0)$ and $\alpha(\theta, t_1)$ intersect if $u(t_0) - u(t_1)$ and $v(t_0) - v(t_1)$ have different signs. But this is impossible because $\dot{u}$ and $\dot{v}$ have the same sign. Denote by $u_\pm$ the supremum and the infimum of u, and by $v_\pm$ the supremum and infimum of v. The image of α is the half ellipse with semi-axes (u_+, v_+) minus the smaller ellipse with semi-axes (u_-, v_-) (see Fig. 5.5), which implies the last statement. $\square$

In particular, if the infimum of $\dot{a}\dot{b}$ is zero, we get a global fibration in the sense of Definition 3.1.5 by Corollary 5.3.10. Note that this is always the case, when the G_2-structure defined by Foscolo–Haskins–Nordström extend to the singular orbit $\mathrm{SU}(2) \times \mathrm{SU}(2)/K$ (cfr. Section 2.2.5.3).

On the other hand, if the infimum of $\dot{a}\dot{b}$ is not zero, we can still describe the $\mathbb{T}^2$-invariant associatives splitting $B \cong (0, \pi) \times \mathrm{Int}(I)$ into $(0, \pi/2) \times \mathrm{Int}(I)$ and $(\pi/2, \pi) \times \mathrm{Int}(I)$.

We summarize everything in the following theorem.

Theorem 5.5.5 (Aslan–T. [6]; $\mathbb{T}^2$-invariant associatives in FHN manifolds)**.** *Consider the stratification, as given in Section 5.1.1, of the FHN manifolds into* $M_P \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$ *with respect to the* $\mathbb{T}^2 \times \mathrm{SU}(2)$*-action.*

We first consider the subset $((\mathrm{SU}(2) \times \mathrm{SU}(2))/K_0) \times \mathrm{Int}(I)$*, which does not intersect* $\mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4$*. Then, each strata decomposes into* $\mathbb{T}^2$*-invariant associatives in the following way:*

- M_P *is fibred by* $\mathbb{T}^2$*-invariant associatives which are horizontal lifts of level sets of* $|\mu| = 4\dot{a}\dot{b}\sin\theta$ *in* $B \cong (0,\pi)\times\mathrm{Int}(I)$*, where* θ *is determined by* $\cos\theta = \langle v, w\rangle$ *and* v, w *are images of the Hopf maps:* $(v = q\bar{p}ip\bar{q}, w = qi\bar{q}) \in S^2 \times S^2$*. The topology of these associatives is* $\mathbb{T}^2 \times \mathbb{R}$*. If the* G_2*-structure extends smoothly to* $(\mathrm{SU}(2) \times \mathrm{SU}(2))/K$*, these associatives do not intersect* $(\mathrm{SU}(2) \times \mathrm{SU}(2))/K$*.*

- *As in Proposition 5.5.3,* $\mathcal{S}_1$ *admits a submersion over* $S^2 \cup S^2$ *with totally geodesic* $\mathbb{T}^2$*-invariant associative fibres of topology* $\mathbb{T}^2 \times \mathbb{R}$*. If the* G_2*-structure extends smoothly to* $(\mathrm{SU}(2)\times\mathrm{SU}(2))/K$*, these associatives extend smoothly to associatives of topology* $S^1 \times \mathbb{R}^2$ *in* M*.*

When the G_2*-structure extends to* $\mathrm{SU}(2) \times \mathrm{SU}(2)/K$*, we distinguish two cases:*

- *If* $K = \Delta\,\mathrm{SU}(2)$ *or* $K = \mathrm{Id}_{\{\mathrm{SU}(2)\}} \times \mathrm{SU}(2)$*, then,* $\mathrm{SU}(2)\times\mathrm{SU}(2)/K$ *is a* $\mathbb{T}^2$*-invariant associative of topology* S^3 *as it is* $\mathcal{S}_2$ *or* $\mathcal{S}_3 \cup \mathcal{S}_4$*.*

- *If* $K = K_{m,n}$*, the set consists of* $\mathcal{S}_1$ *and* $\mathcal{S}_2$*. There exists a submersion over* S^2 *with* $\mathbb{T}^2$*-invariant associative fibres of topology* $L(n : m, -n)$*. Moreover, there are two additional* $\mathbb{T}^2$*-invariant associatives corresponding to the two connected components of* $\mathcal{S}_2$*.*

5.5.2.5 $\mathbb{T}^3$-invariant coassociatives

Let $\mathbb{T}^3$ be the torus generated by V_1, U_1, U_2. It is straightforward to see that the singular set of this action, $\overline{\mathcal{S}}$, restricted to $((\mathrm{SU}(2) \times \mathrm{SU}(2))/K_0) \times \mathrm{Int}(I)$ is:

$$\overline{S}_P = \{([p,q],t) \in (\mathrm{SU}(2) \times \mathrm{SU}(2)/K_0) \times \mathrm{Int}(I) : p, q \in (\mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}) \subset \mathrm{Sp}(1)\},$$

which is contained in $\subset \mathcal{S}_+ \cup \mathcal{S}_-$. On $\overline{\mathcal{S}}_P$ the stabilizer is 1-dimensional and it is mapped, via $\pi_{\mathbb{T}^2}$ to $\{(\pm i, \pm i, t), (\pm i, \mp i, t)\}$.

On $\mathrm{SU}(2) \times \mathrm{SU}(2)/K$, with $K = \Delta\,\mathrm{SU}(2)$ or $K = \{1\} \times \mathrm{SU}(2)$, the stabilizer is everywhere 1-dimensional apart from the intersection of $\mathrm{SU}(2)\times\mathrm{SU}(2)/K$ with the closure of $\overline{S}_P$, where the stabilizer is 2-dimensional. If $K = K_{m,n}$, the stabilizer at $[(p,q)] \in$

$(\mathrm{SU}(2)\times\mathrm{SU}(2))/K_{m,n}$ is 2-dimensional if p and q are in $\mathbb{C}\times\{0\}\cup\{0\}\times\mathbb{C}$, it is 1-dimensional if p or q is in $\mathbb{C}\times\{0\}\cup\{0\}\times\mathbb{C}$ and it is 0-dimensional otherwise.

By Proposition 5.4.3, the $\mathbb{T}^3$-invariant coassociatives, in $M\setminus\overline{\mathcal{S}}$, are the level sets of the map $(\theta_1^1,\theta_1^2,\nu)$:

$$([p,q],t)\mapsto\left(2av_1-2(a-b)\langle v,w\rangle_{\mathbb{R}^3}w_1,-2(b+c_2)w_1,-4(b-c_1)\langle v,w\rangle_{\mathbb{R}^3}\right),$$

where v,w are as above.

We now characterize the $\mathbb{T}^3$-invariant coassociatives intersecting the 1-dimensional and the 2-dimensional stabilizer.

Given $([p,q],t_0)\in\overline{\mathcal{S}}_P$, it is mapped via $(\theta_1^1,\theta_1^2,\nu)$ to $(\epsilon_1 2b(t_0),\epsilon_2 2(b(t_0)+c_2),\epsilon_3 4(b(t_0)-c_1))$, where $\epsilon_i\in\{0,1\}$ take one of four possibilities for which $\epsilon_1\epsilon_2\epsilon_3=1$, depending whether p and q are in $\mathbb{C}\times\{0\}$ or $\{0\}\times\mathbb{C}$. We now turn our attention to $\mathrm{SU}(2)\times\mathrm{SU}(2)/K$.

Case 1 ($K=\Delta\,\mathrm{SU}(2)$). If $K=\Delta\,\mathrm{SU}(2)$, a $\mathbb{T}^3$-invariant coassociative intersects the set with 1-dimensional stabilizer in $\mathrm{SU}(2)\times\mathrm{SU}(2)/K$, if and only if it is the preimage of $(x,0,0)$ for $x\in(-2c_1,2c_1)$. It intersects the set with 2-dimensional stabilizer, and hence singular by Theorem 5.4.5, if and only if $x=\pm 2c_1$.

Case 2 ($K=\{1_{\mathrm{SU}(2)}\}\times\mathrm{SU}(2)$). In this case, the $\mathbb{T}^3$-invariant coassociatives corresponding to the preimages of $(0,0,x)$, for $x\in[-4c_1,4c_1]$, are the ones intersecting $\mathrm{SU}(2)\times\mathrm{SU}(2)/K$. Among them, the one intersecting the set with 2-dimensional stabilizer are the preimages of $(0,0,\pm 4c_1)$.

Case 3 ($K=K_{m,n}$). When $K=K_{m,n}$, the coassociatives intersecting the set with 0-dimensional stabilizer in $\mathrm{SU}(2)\times\mathrm{SU}(2)/K$ are the the level sets of points in:

$$\left\{(2mnr_0^3xy,-2n(m+n)r_0^3y,-4m(m+n)r_0^3x):x,y\in(-1,1)\right\};$$

they intersect the set with 1-dimensional stabilizer they are the level set of points in:

$$\left\{(2mnr_0^3xy,-2n(m+n)r_0^3y,-4m(m+n)r_0^3x):x=\pm1,y\in(-1,1)\text{ or }y=\pm1,x\in(-1,1)\right\};$$

and they are singular if they are the preimage of:

$$(\pm 2mnr_0^3,-2n(m+n)r_0^3,\mp 4m(m+n){r_0}^3)\text{ or }(\pm 2mnr_0^3,+2n(m+n)r_0^3,\pm 4m(m+n){r_0}^3).$$

In particular, from this discussion one could characterize the $\mathbb{T}^3$-invariant coassociatives of different topology (see Section 5.5.3.4 for an explicit example). Note that, the only topological possibilities are the $\mathbb{T}^3\times\mathbb{R}$, $\mathbb{T}^2\times\mathbb{R}^2$ and the singular ones $\mathbb{T}^2\times\mathbb{R}\times\mathbb{R}^+$.

5.5.2.6 SU(2)-invariant coassociatives

Finally, we study SU(2)-invariant coassociatives. Similarly to Section 5.5.2.2, we can compute $\varphi(V_1, V_2, V_3) = c_2$. Hence, there are SU(2)-invariant coassociatives if and only if $c_2 = 0$. If this is the case, the coassociative submanifolds are of the form:

$$\{([p_0, q], t) \in ((\mathrm{SU}(2) \times \mathrm{SU}(2))/K_0) \times \mathrm{Int}(I) : q \in \mathrm{SU}(2), t \in \mathrm{Int}(I)\},$$

for every fixed $p_0 \in \mathrm{SU}(2)$. As we assumed $c_2 = 0$, the only possibility to extend the G_2-structure to $\mathrm{SU}(2) \times \mathrm{SU}(2)/K$ is for K equal to $\{1\} \times \mathrm{SU}(2)$. In this situation, the resulting SU(2)-invariant coassociatives extend to smooth $\mathbb{R}^4$s.

5.5.3 Bryant–Salamon manifold

As an explicit special case of Section 5.5.2, we consider the Bryant–Salamon manifolds of topology $S^3 \times \mathbb{R}^4 = \{(x, a) \in \mathbb{H}^2 : ||x|| = 1\}$. Up to an element of the automorphism group, we can restrict ourselves to the following actions:

1. $(\lambda_1, \lambda_2, \gamma)(x, a) \mapsto (\lambda_1 x\bar{\gamma}, \lambda_2 a\bar{\gamma})$,
2. $(\lambda_1, \lambda_2, \gamma)(x, a) \mapsto (\lambda_1 x\overline{\lambda_2}, \gamma a\overline{\lambda_2})$,
3. $(\lambda_1, \lambda_2, \gamma)(x, a) \mapsto (\gamma x\overline{\lambda_2}, \lambda_1 a\overline{\lambda_2})$,

where $(\lambda_1, \lambda_2, \gamma) \in \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{Sp}(1)$ and the U(1)s are generated by quaternionic multiplication by i. Note that Case (1) can be reconducted to the discussion in Section 5.5.2, picking $K = \Delta\,\mathrm{SU}(2)$ and up to a change of variables, while Case (2) and Case (3) picking $K = \{1\} \times \mathrm{SU}(2)$. However, to be more explicit, we fix the description of the Bryant–Salamon manifold as in Eq. (2.2.9) and we adjust the arguments of Section 5.5.2 accordingly.

5.5.3.1 The stratification

We first notice that the principal stabilizer is generated by $(-1, -1) \in \mathbb{T}^2 \times \mathrm{SU}(2)$ for all cases, hence $G^{\mathrm{SU}(2)} = \mathrm{SO}(3)$.

The stratification for Case (1) is:

$$\begin{aligned}
M_P = (S^3 \times \mathbb{H}^*) \setminus \mathcal{S}_1, \quad & \mathcal{S}_1 = \{(x, a) \in S^3 \times \mathbb{H}^* : \overline{x}a \in \mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}\}, \\
\mathcal{S}_2 = \{(x, 0) \in \mathbb{H}^2\}, \quad & \mathcal{S}_3 = \emptyset, \quad \mathcal{S}_4 = \emptyset,
\end{aligned}$$

for Case (2) it is:

$$\begin{aligned} &M_P = (S^3 \times \mathbb{H}^*) \setminus \mathcal{S}_1, \quad \mathcal{S}_1 = \{(x,a) \in \mathbb{H}^2 : x \in \mathrm{U}(1) \times \{0\} \cup \{0\} \times \mathrm{U}(1)\}, \\ &\mathcal{S}_2 = \emptyset, \quad \mathcal{S}_3 = \{(x,0) \in \mathbb{H}^2\} \setminus \mathcal{S}_1, \quad \mathcal{S}_4 = \{(x,0) \in \mathbb{H}^2\} \cap \mathcal{S}_1, \end{aligned}$$

finally, for Case (3) it is:

$$\begin{aligned} &M_P = (S^3 \times \mathbb{H}^* \setminus \mathcal{S}_1), \quad \mathcal{S}_1 = \{(x,a) \in \mathbb{H}^2 : a \in \mathrm{U}(1) \times \{0\} \cup \{0\} \times \mathrm{U}(1)\}, \\ &\mathcal{S}_2 = \{(x,0) \in \mathbb{H}^2\} \quad \mathcal{S}_3 = \mathcal{S}_4 = \emptyset. \end{aligned}$$

5.5.3.2 The multi-moment maps

Before computing the multi-moment maps, we write the explicit form of the projection to the $\mathbb{T}^2$-quotient: $\pi_{\mathbb{T}^2}$. In $S^3 \times \mathbb{H}^*$, these take the form:

$$\pi_{\mathbb{T}^2} : S^3 \times S^3 \times \mathbb{R}^+ \to S^2 \times S^2 \times \mathbb{R}^+ \quad (p,q,r) \mapsto (v,w,r),$$

where, for Case (1) $v = \overline{p}ip, w = \overline{q}iq$, for Case (2) $v = q\bar{p}ip\bar{q}, w = qi\bar{q}$ and, for Case (3), $v = pi\bar{p}, w = p\bar{q}iq\bar{p}$. The multi-moment maps, which pass to the $\mathbb{T}^2$-quotients, are:

	Case (1)	Case (2)	Case (3)
ν	$2\sqrt{3}r^2\langle v,w\rangle_{\mathbb{R}^3}$	$-\frac{\sqrt{3}}{2}(3c+4r^2)\langle v,w\rangle_{\mathbb{R}^3}$	$-2\sqrt{3}r^2\langle v,w\rangle_{\mathbb{R}^3}$
θ^1	$\frac{\sqrt{3}}{4}(3c+4r^2)v$	$\sqrt{3}r^2 v$	$\frac{\sqrt{3}}{4}(3c+4r^2)v$
θ^2	$-\sqrt{3}r^2 w$	$-\sqrt{3}r^2 w$	$-\sqrt{3}r^2 w$
θ^3	$-3r^2(c+r^2)^{1/3} v \times_{\mathbb{R}^3} w$	$-3r^2(c+r^2)^{1/3} v \times_{\mathbb{R}^3} w$	$3r^2(c+r^2)^{1/3} v \times_{\mathbb{R}^3} w$

.

5.5.3.3 $\mathbb{T}^2$-invariant associatives

The description of the $\mathbb{T}^2$-invariant associatives follows exactly as in the FHN manifolds. For instance, we obtain the following result for Case (1).

Theorem 5.5.6 (Aslan–T. [6]; $\mathbb{T}^2$-invariant associatives in Bryant–Salamon manifolds)**.** *Consider the stratification, as given in Section 5.1.1, of the Bryant–Salamon space into $M_P \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3 \cup \mathcal{S}_4$ with respect to the $\mathbb{T}^2 \times \mathrm{SU}(2)$-action of Case (1). Then, each strata decomposes into $\mathbb{T}^2$-invariant associatives in the following way:*

- *M_P is fibred by $\mathbb{T}^2$-invariant associatives which are horizontal lifts of level sets of $|\mu| = 3r^2(c+r^2)^{1/3}\sin\theta$ in $B \cong (0,\pi) \times \mathbb{R}^+$, where θ is determined by $\cos\theta = \langle v,w\rangle$ and v,w are images of the Hopf maps: $(v = pi\bar{p}, w = qi\bar{q}) \in S^2 \times S^2$. The topology of these associatives is $\mathbb{T}^2 \times \mathbb{R}$ and they do not intersect the zero section.*

- $\mathcal{S}_1$ *admits a fibration over* $S^2 \cup S^2$ *with totally geodesic* $\mathbb{T}^2$*-invariant associative fibres of topology* $\mathbb{T}^2 \times \mathbb{R}$*. These associatives extend smoothly to associatives of topology* $S^1 \times \mathbb{R}^2$ *in* M*.*

- $\mathcal{S}_2$ *is the zero section, which is an associative totally geodesic group orbit of topology* S^3*.*

- $\mathcal{S}_3 = \mathcal{S}_4 = \emptyset$*.*

Remark 5.5.7. The associatives of topology $S^1 \times \mathbb{R}^2$ were independently constructed by Fowdar in [33]. The author also constructed a similar family in the BGGG manifolds.

5.5.3.4 $\mathbb{T}^3$-invariant coassociatives

Up to an element of the autormorphism group, we can choose, for all the three cases, the torus $\mathbb{T}^3$ acting on $(x, a) \in S^3 \times \mathbb{R}^4$ as follows:

$$(\lambda_1, \lambda_2, \lambda_3)(x, a) \mapsto (\lambda_1 x \bar{\lambda}_3, \lambda_2 a \bar{\lambda}_3),$$

where all the λ_is are generated by multiplication by i.

It is straightforward to see that the singular set of this action, $\overline{\mathcal{S}}$, is given by the zero section and the following subset:

$$\overline{S}_P = \left\{(x, a) \in S^3 \times \mathbb{H} : x, a \in (\mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}) \subset \mathbb{C} \times \mathbb{C}\right\},$$

In the singular set, the stabilizer is everywhere 1-dimensional apart from the points in:

$$\left\{(x, 0) \in S^3 \times \mathbb{H} : x \in (\mathbb{C} \times \{0\} \cup \{0\} \times \mathbb{C}) \subset \mathbb{C} \times \mathbb{C}\right\},$$

where the stabilizer is 2-dimensional.

By Proposition 5.4.3, the $\mathbb{T}^3$-invariant coassociatives are given by the level sets of the map $(\theta_1^1, \theta_1^2, \nu)$, which is explicitly given by:

$$(p, q, r) \mapsto \left(\frac{\sqrt{3}}{4}(3c + 4r^2)v_1, -\sqrt{3}r^2 w_1, 2\sqrt{3}r^2 \langle v, w \rangle_{\mathbb{R}^3}\right),$$

where $v, w \in S^2 \subset \mathbb{R}^3$ are defined accordingly to (1). By Theorem 5.4.5, the $\mathbb{T}^3$-invariant coassociatives are smooth topological $\mathbb{T}^3 \times \mathbb{R}$, apart from the ones intersecting the points with one or 2-dimensional stabilizer, which are smooth $\mathbb{T}^2 \times \mathbb{R}^2$s and $\mathbb{T}^2 \times \mathbb{R} \times \mathbb{R}^+$ cones, respectively. The intersection with the 2-dimensional stabilizer occurs only to the preimages of $\{(\pm\frac{3\sqrt{3}}{4}c, 0, 0)\}$. The $\mathbb{T}^3$-invariant coassociatives intersecting

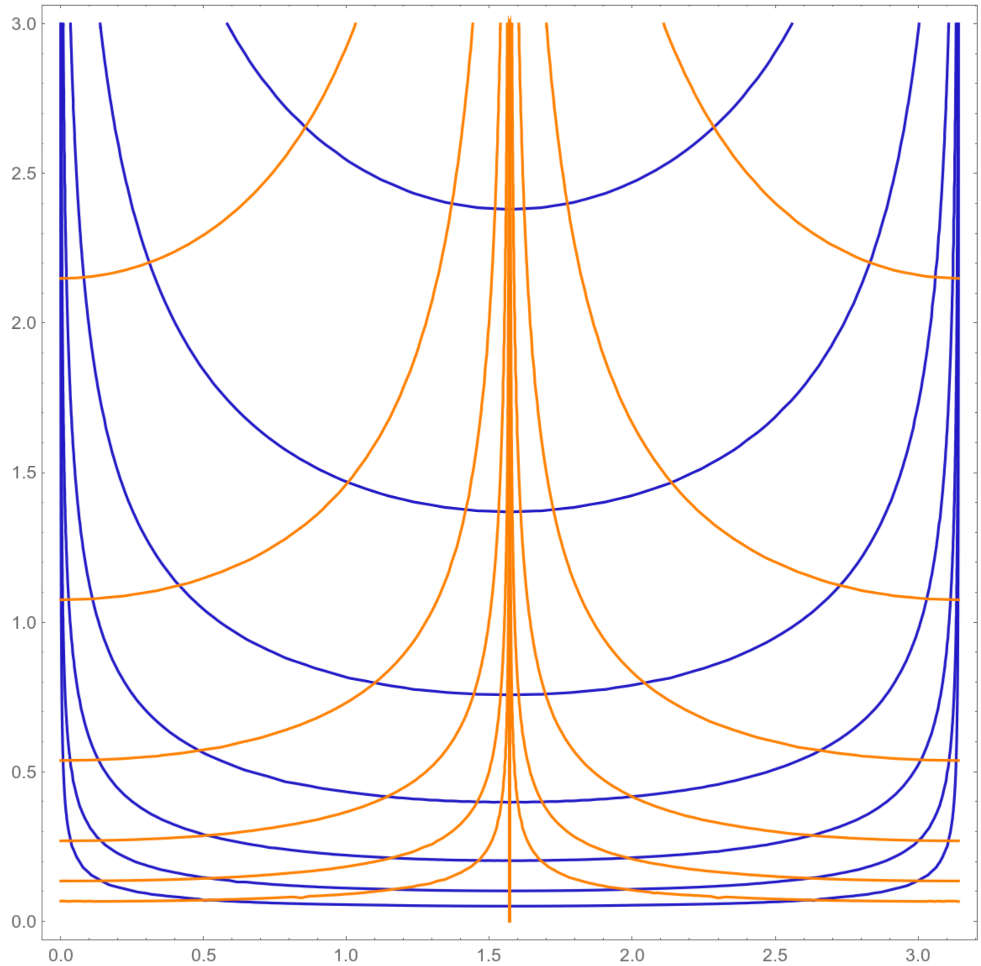


Figure 5.6: Blue: The level sets of $|\mu| = 3r^2(c+r^2)^{1/3}\sin\theta$ in $B = (0,\pi)\times\mathbb{R}^+$. Every level set represents an $\mathrm{SU}(2)$-family of $\mathbb{T}^2$-invariant associatives in M_P. Orange: The level sets of $\nu = 2\sqrt{3}r^2\cos\theta$. Every level set represents an S^2 family of $\mathbb{T}^3$-invariant coassociatives in M_P. The vertical line represents the ones intersecting the zero section, two of these $\mathbb{T}^3$-invariant coassociatives are singular.

the 1-dimensional stabilizer are the ones corresponding to the fibres of the following set: $\{(x,0,0) : x \in (-\frac{3\sqrt{3}c}{4}, \frac{3\sqrt{3}c}{4})\} \cup A$, where A is:

$$\left\{\left(\pm\left(\frac{3\sqrt{3}c}{4}+a\right), -a, \pm 2a\right) : a \in \mathbb{R}^+\right\} \cup \left\{\left(\pm\left(\frac{3\sqrt{3}c}{4}+a\right), +a, \mp 2a\right) : a \in \mathbb{R}^+\right\}$$

5.5.3.5 SU(2)-invariant coassociatives

One can compute $\varphi_c(V_1, V_2, V_3)$ for Case (1), Case (2) and (3). This vanishes only when $c = 0$ in Case (1) and Case (3), while for Case (2) it is always vanishing. We deduce that $\mathrm{SU}(2)$-invariant coassociatives are given by fibres of the standard projection to S^3

(cfr. [49, Section 4]).

5.5.3.6 Another family of associative submanifolds

The associatives fibres of $\mathcal{S}_1 \to S^2$ in Theorem 5.5.6 are products of a plane in $\mathbb{R}^4$ times a geodesic in S^3. In general, one can take any 2-dimensional vector subspace $W \subset \mathbb{R}^4$, spanned by the orthonormal vectors w_1, w_2, and observe that $w_1 \times w_2$ is tangent to S^3. For every $p \in S^3$, we can consider $\gamma_{W,p}$ to be the unit length geodesic starting at p with velocity $w_1 \times w_2$, and observe that $\gamma_{W,p} \times W$ is an associative submanifold. These examples are not only part of the family of $\mathbb{T}^2$-invariant associative submanifolds, but also of the following family, where each associative contains an affine plane $\bar{W} := W \oplus x$ in $\mathbb{R}^4$. Here, W is a 2-dimensional vector subspace of $\mathbb{R}^4$ and x is in the Euclidean perpendicular subspace $W^\perp$. The orthogonal complement $W^\perp$ carries a unique positive complex structure, so we can define the curve contained in it:

$$\delta_{W,x}(t) = e^{-i\frac{t}{2}} x.$$

Proposition 5.5.8. *Let p be a point in S^3, $\bar{W} = W \oplus x$ be an affine plane with $x \in W^\perp$. The unique associative containing $\{p\} \times \bar{W}$ is*

$$N := \{(\gamma_{W,p}(t), y, \delta_{W,x}(t)) \in S^3 \times W \times W^\perp \mid y \in W, t \in \mathbb{R}\}.$$

Proof. As the uniqueness follows immediately from the local existence and uniqueness theorem, we only need to prove that N is an associative submanifold. We use the parametrisation of $S^3 \times \mathbb{R}^4$ as in Section 2.2.4.2. By applying elememts of the automorphism group $\mathrm{SU}(2)^3$, we can assume without loss of generality that $W = \{a_2 = a_3 = 0\}$. Moreover, we choose a left-invariant frame $\{E_1, E_2, E_3\}$ on S^3 such that the tangent space of N is spanned by $\{\partial_{a_0}, \partial_{a_1}, e_1 - (a_3\partial_{a_2} - a_2\partial_{a_3})/2\}$ at any point of N. We conclude as $*\varphi(e_1 - (a_3\partial_{a_2} - a_2\partial_{a_3})/2, \partial_{a_0}, \partial_{a_1}, \cdot) = 0$ at any point of N. $\square$

In particular, Proposition 5.5.8 extends the description of possibly twisted calibrated subbundles in manifolds of exceptional holonomy which was started by Karigiannis, Leung and Min-Oo in [48, 50].

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